The Projection Settlement

Technical Note for Stage v20 of the FDT Bell Simulation

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Abstract

This note documents the stand-alone edition of stage v20, the last stage of the helix-projection Bell simulation of Fundamental Density Theory (FDT). The stage keeps two local sign records on one shared phase and supplies the measure of that phase: the projection measure of the clamp that hosts the pair's settlement, $p_{\mathrm{host}}(\varphi \mid a) = |\cos(\varphi - a)|/4$ (core I-4). On it the records return $E = -\cos\Delta$ on every record, with both single-wing marginals 1/2 and the same table whichever clamp hosts the settlement: $S = 2.8276 \pm 0.0014$ with all 4×10^6 records kept, against 2.0002 ± 0.0017 for the clamp-independent control. The nine transport scenarios and the nine-outcome detector tables equal the Born layer cell by cell. The measure is set by a clamp, and the price is exact: Hall's measure $M_1 = 2\sqrt{2} - 2$, at which the one-sided relaxed bound $S \le 2 + M_1$ is met with equality and $2\sqrt{2}$ is the ceiling. The note states the model, the algorithms, the eight blocks of the stage with their reference numbers, the outputs, the eleven decision checks and thirty-five tests, and the commands that reproduce every number on a laptop in under a minute.

Contents

1 The package **2**

1.1 What it is . . . 2

1.2 Contents . . . 3

1.3 Quick start . . . 3

2 The model **3**

2.1 Chart and records . . . 3

2.2 The two measures of the shared phase . . . 4

2.3 The kernel . . . 5

2.4 CHSH and the arithmetic on one phase . . . 5

2.5 Transport and ensembles . . . 6

2.6 The sphere of the helix chart . . . 6

2.7 The detector layer . . . 6

3 The price of the clamp route **7**

*An AI model made by Anthropic. Its part in the work is stated under "Author contributions".

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4 Implementation **8**
4.1 Module map 8
4.2 Exact arc integration 8
4.3 Sampling the settlement 9
4.4 Seeds and streams 9
4.5 Detector tables and counts 9
4.6 Quadrature, tolerances and errors 9

5 The stage, block by block **10**

6 Outputs **11**

7 Verification **13**
7.1 Decision checks 13
7.2 Tests 13
7.3 Reproducibility 13

8 Using the readout as a library **14**

9 Command line **15**

10 Status and scope **16**

11 Convergence statement **16**

1 The package

1.1 What it is

The helix-projection Bell simulation is an archive of stages; each stage is a runner with its tests, its results and its document. Stage v20, the projection settlement, is the last, and it is the stage that returns the cosine kernel on every record. This edition holds stage v20 alone, arranged to run by itself: the readout library, the stage runner, its thirty-five tests, the nine library modules the runner imports, the reference outputs, a comparison tool, two examples and this note.

The runners, tests, results and documents of the earlier stages are not part of this edition. The author supplies the complete archive on request. The nine carried modules are byte-identical to their copies in the complete archive, and their file headers describe each module's chart as written for the stage that introduced it. The two v20 modules and this note state the status of every item under the locked core, v6.7 [2].

The companion paper [1] gives the physics in full: the helix chart, the transport of the shared bivector grade, the fluctuating-media scenarios, the settlement, and the reading of the public experimental record. This note is the engineering side of its section on the projection settlement.

1.2 Contents

Path	Content
`README.md`	Quick start and file list
`LICENSE`	MIT License for the software
`MANIFEST.sha256`	SHA-256 of every file of the package
`requirements.txt`	NumPy, SciPy, Matplotlib
`run.sh`, `run_tests.sh`	The stage and its tests
`simulation/fdt_settlement_readout.py`	The readout library (NumPy only)
`simulation/fdt_v20_projection_settlement.py`	The stage: eight blocks, decision checks, report, ledger, figure
`simulation/test_v20_projection_settlement.py`	Thirty-five tests
`simulation/` (nine further modules)	The transport chain and the Born-layer comparison tables, carried unchanged (Table 1)
`reference/`	Report, ledger and figure of the default run
`tools/compare_to_reference.py`	Compares a fresh run with `reference/`
`examples/quickstart.py`	The two measures on one million records per setting
`examples/price_of_the_clamp_route.py`	$M_1(\theta)$ and the bound at 181 clamp separations
`docs/`	This note, source and PDF

1.3 Quick start

The stage needs Python 3.13 or later with NumPy, SciPy and Matplotlib.

```
python3 -m pip install -r requirements.txt
./run_tests.sh                          # 35 tests
./run.sh                                # the stage; writes ./output
python3 tools/compare_to_reference.py   # ./output against ./reference
```

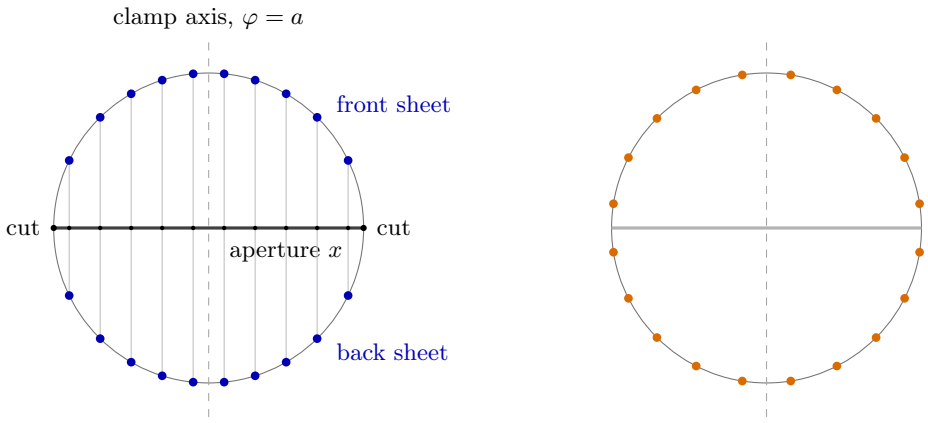
The tests end with `Ran 35 tests` and `OK`. The stage prints its report and ends with `INTERNAL IMPLEMENTATION AUDIT: PASS`; its exit status is nonzero when a decision check fails. The comparison ends with `RESULT : MATCH`. On an Apple-silicon laptop the stage takes 32 s and the tests 18 s; on the Linux x86-64 reference machine, 55 s and 30 s. `./run.sh --quick` runs the same blocks on smaller samples. The variable `PYTHON_BIN` selects the interpreter, as in `PYTHON_BIN=python3.14 ./run.sh`. Section 9 lists every option.

2 The model

2.1 Chart and records

The helix chart of the stage is the doubled-angle circle. A pair carries one shared phase φ; the partner is carried at $\varphi + \pi$, the shared bivector grade in the opposite sense. A clamp is an analyzer, a boundary condition of the settlement (P-2). Clamp angles are doubled angles: a linear analyzer at laboratory angle θ is the clamp $a = 2\theta$, the Stokes axis $(\sin a, 0, \cos a)$. Each arm's flight through its medium is a real proper Jones rotation by an angle β, which enters the chart as 2β. Each wing keeps one record,

$$A = \mathrm{sgn}_+ \cos(\varphi + 2\beta_A - a), \qquad B = \mathrm{sgn}_+ \cos(\varphi + \pi + 2\beta_B - b), \tag{1}$$



$|\cos(\varphi - a)|/4$: uniform in x, either sheet $1/2\pi$: uniform in φ, blind to the clamp

Figure 1: The two measures of the shared phase on the doubled-angle circle. Left: the settlement lands uniformly across the projected aperture of the host clamp and is lifted to the circle on the front or the back sheet; the phase crowds toward the clamp axis and thins to zero at the cut. Right: the clamp-independent control. The records of Eq. (1) are read on both.

with $\mathrm{sgn}_+(x) = +1$ for $x > 0$ and -1 otherwise. A's record reads A's clamp, A's transport and the phase that reaches A; B's record reads B's. The two record formulas are the same for every measure of the phase that the stage runs.

2.2 The two measures of the shared phase

The clamp at a is the cut $\cos(\varphi - a) = 0$, and the coordinate across the cut is the projected one, $x = \sin(\varphi - a) \in [-1, 1]$. I-4 reads the wavefunction as the projection of the helix onto the plane normal to its axis, and a clamp reads the helix in that projection. The pair and its two clamps are one system, and the pair's record is one settlement of it (0.1a), its mode geometry set by the clamps (P-2). For the equal-weight pair the settlement lands uniformly across the projected aperture of the clamp that hosts it, x uniform on each of the two sheets (I-4, core v6.7):

$$\boxed{p_{\text{host}}(\varphi \mid a) = \frac{|\cos(\varphi - a)|}{4}} \qquad p(\varphi_A, \varphi_B \mid a, b) = \delta(\varphi_B - \varphi_A - \pi)\, p_{\text{host}}(\varphi_A \mid \text{host clamp}). \quad (2)$$

With transport the measure is $|\cos u|/4$ in the host's impact phase, $u_A = \varphi + 2\beta_A - a$ or $u_B = \varphi + \pi + 2\beta_B - b$, and the shared phase is its pull-back through that arm's rotation. The stage runs three host rules: host A, host B, and a host drawn per trial with weight w for A (`host="either"`, $w = 1/2$ by default).

The control is the clamp-independent measure, φ uniform on the circle with density $1/2\pi$. Uniform in the projected coordinate is the projection measure; uniform in the angle is the control; the two differ by the change of variable $\varphi - a = \arcsin x$ (Fig. 1).

2.3 The kernel

Theorem 1 (Cosine kernel of the settlement)**.** *Let* $\Delta = a - b - 2(\beta_A - \beta_B)$*. On the measure of Eq.* (2) *the records of Eq.* (1) *satisfy*

$$P(A = B) = \frac{1 - \cos\Delta}{2}, \qquad E(\Delta) = -\cos\Delta, \qquad P(A = +1) = P(B = +1) = \frac{1}{2}, \tag{3}$$

for either host clamp and for every mixture of hosts.

Proof. Take host A. In its impact phase $A = \operatorname{sgn}_+ \cos u$ and $B = -\operatorname{sgn}_+ \cos(u + \Delta)$ almost everywhere. The host's own record removes the absolute value of the measure, $|\cos u| \operatorname{sgn}_+ \cos u = \cos u$, so

$$\begin{aligned} E &= -\frac{1}{4}\int_0^{2\pi} \cos u \, \operatorname{sgn}_+ \cos(u + \Delta)\, du = -\frac{1}{4}\int_0^{2\pi} \cos(v - \Delta) \, \operatorname{sgn}_+ \cos v \, dv \\ &= -\frac{\cos\Delta}{4}\int_0^{2\pi} |\cos v|\, dv = -\cos\Delta, \end{aligned} \tag{4}$$

the $\sin\Delta$ term vanishing by symmetry. The measure is invariant under $u \mapsto u + \pi$, which reverses both records, so each record is ± 1 with probability $1/2$ at every clamp pair. Host B is the same computation in u_B with $\Delta \mapsto -\Delta$, and a mixture of hosts is a mixture of two equal tables. □

The same computation with the uniform measure gives $P(A = B) = |\Delta|/\pi$ on $|\Delta| \le \pi$, the triangle

$$E_{\text{uni}}(\Delta) = -1 + \frac{2|\Delta|}{\pi} = -\frac{2}{\pi}\arcsin\cos\Delta. \tag{5}$$

The two kernels are the two sides of the arcsine law: the uniform angle returns the arcsine rounding of the cosine, the projection measure returns the cosine. Each wing's record reads its own clamp and the phase it receives; no single-wing record carries a reading of the remote clamp, and no order of the two records enters the statistics.

2.4 CHSH and the arithmetic on one phase

The stage uses the facet $S = E_{00} + E_{01} + E_{10} - E_{11}$ at the clamps

$$a = (0, \tfrac{\pi}{2}), \qquad b = (-\tfrac{3\pi}{4}, \tfrac{3\pi}{4}), \tag{6}$$

the linear analyzers $0, \pi/4$ and $\mp 3\pi/8$. The settlement gives $S = 2\sqrt{2}$ there and the uniform measure $S = 2$; the ratio is $\sqrt{2} = K_G(2)$, the Grothendieck constant of order two [9].

Reading all four clamp positions on one phase gives

$$A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 = A_0(B_0 + B_1) + A_1(B_0 - B_1) = \pm 2, \tag{7}$$

since one bracket is ± 2 and the other is 0. At the clamps of Eq. (6) the value is $+2$ at every phase. The identity holds phase by phase under both measures. The uniform control averages it over one measure and stays at 2. The settlement's four correlators are averages over the measures of the two host settings, and the bound whose premise is $p(\lambda \mid a, b) = p(\lambda)$ does not constrain them. Section 3 states what does.

2.5 Transport and ensembles

The transport enters the kernel only through the relative angle $\Delta = a - b - 2(\beta_A - \beta_B)$. A scenario is an ensemble of N histories j with rotations $(\beta_{A,j}, \beta_{B,j})$; its port table is the history average of the single-history tables, and with $\gamma = \langle e^{2i(\beta_A - \beta_B)} \rangle$ the kernel is $E = -\operatorname{Re}\big[\gamma^* e^{i(a-b)}\big]$, so that $S_{\text{fixed}} = 2\sqrt{2}\operatorname{Re}\gamma$ at the clamps of Eq. (6). Matched histories ($\beta_A = \beta_B$) return $2\sqrt{2}$; unresolved relative histories lower $|\gamma|$.

The histories themselves come from the transport chain carried from stage v19. Each arm crosses a static synthetic medium described by the laboratory-medium aperture token $\alpha_n = 1 - n^{-2}$, drawn as a Gaussian field in $q = \operatorname{logit}\alpha_n$ with a baseline, optional gradient patches, a global offset and eighteen Fourier modes of correlation length 0.18 m. The index is the aperture assignment $n_{\text{cone}}(\alpha_n) = (1-\alpha_n)^{-1/2}$ of P-4, so $\nabla \ln n = \alpha_n \nabla q/2$. A ray of path length 1 m and its transverse frame follow Fermat's principle,

$$\mathbf{r}' = \mathbf{t}, \qquad \boldsymbol{\Omega} = \mathbf{t} \times \nabla \ln n, \qquad R' = \tfrac{1}{2}(0, \boldsymbol{\Omega})R, \tag{8}$$

integrated by a second-order midpoint Lie step of 2 mm. The Jones map $J = V_{\text{out}}^T Q V_{\text{in}}$ of the transported frame is a real proper rotation, and $\beta = \operatorname{atan2}(J_{10}, J_{00})$. The nine scenarios differ in the cross-arm correlation of the field coefficients, the baselines and the order of the gradient patches (Table 3).

2.6 The sphere of the helix chart

For analyzer axes off one great circle the measure is the disc measure: $\boldsymbol{\lambda}$ uniform on the unit disc normal to the settled axis $\mathbf{s}$, the projection plane of I-4, lifted to the sphere on either sheet,

$$p(\boldsymbol{\lambda} \mid \mathbf{s}) = \frac{|\mathbf{s} \cdot \boldsymbol{\lambda}|}{2\pi}, \qquad A = \operatorname{sgn}_+(\widehat{\mathbf{a}} \cdot \boldsymbol{\lambda}), \quad B = -\operatorname{sgn}_+(\widehat{\mathbf{b}} \cdot \boldsymbol{\lambda}), \qquad E = -\widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}}. \tag{9}$$

The kernel follows as in Eq. (4): with host A, $E = -(2\pi)^{-1}\,\widehat{\mathbf{a}} \cdot \int \boldsymbol{\lambda}\, \operatorname{sgn}_+(\widehat{\mathbf{b}} \cdot \boldsymbol{\lambda})\, d\Omega = -\widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}}$, because $\int \boldsymbol{\lambda}\, \operatorname{sgn}_+(\widehat{\mathbf{b}} \cdot \boldsymbol{\lambda})\, d\Omega = 2\pi\, \widehat{\mathbf{b}}$. This is the one-wing measure of Kochen and Specker [6] and the pair measure of Degorre, Laplante and Roland [7]. The circle measure of Eq. (2) is its great-circle section.

2.7 The detector layer

A local detector turns a wing's sign record, its true port, into one of three outcomes: $+$, $-$ or no click. Efficiency acts on the true port, then label confusion:

$$D = \begin{pmatrix} t\,\eta_+(1-f_+) & t\,\eta_+ f_+ & 1 - t\,\eta_+ \\ t\,\eta_- f_- & t\,\eta_-(1-f_-) & 1 - t\,\eta_- \end{pmatrix}, \tag{10}$$

rows the true port $(+,-)$, columns the record $(+,-,\text{no click})$. t is the collection of that arm on that history. A pair is emitted with probability q. The nine-cell table of a setting pair is

$$C(x,y) = \frac{q}{N} \sum_{j=1}^{N} D_{A,j}^T\, P_j(x,y)\, D_{B,j} + (1-q)\, \mathbf{1}[\text{no click}, \text{no click}], \tag{11}$$

with P_j the 2×2 port table of history j. The stage runs three models at $q = 0.85$: perfect detection; fixed port losses with label confusion ($\eta_\pm = 0.92, 0.87$, $f_\pm = 0.015, 0.008$ at A and $0.90, 0.85$, $0.010, 0.020$ at B); and the same losses with a collection that depends on the arm's own final ray angle ϑ, $t = \exp[-\tfrac{1}{2}(\vartheta/0.65)^2]$. The all-trial CHSH scores no click as -1 on both wings; the coincidence-conditioned CHSH uses the four click–click cells alone.

3 The price of the clamp route

The measure is set by a clamp. In Hall's measure of measurement dependence [3], $M = \sup \int |p_{xy} - p_{x'y'}|\, d\lambda$, with M_1 varying A's clamp at fixed B's and M_2 the reverse, a settlement hosted at A has $M_2 = 0$ and

$$M_1 = \sup_{a,a'} \int_0^{2\pi} \left| p_{\text{host}}(\varphi \mid a) - p_{\text{host}}(\varphi \mid a') \right| d\varphi = 2\sqrt{2} - 2 = 0.828427, \tag{12}$$

the supremum at clamps a quarter turn apart, with $F = 1 - M_1/2 = 2 - \sqrt{2} = 0.585786$.

Proposition 1 (One-sided bound)**.** *For a measure p_x set by A's clamp alone and any records $A_x, B_y \in \{-1, +1\}$ on it,*

$$S = \int p_0 A_0 (B_0 + B_1) + \int p_1 A_1 (B_0 - B_1) \;\le\; 2 + \int (p_0 - p_1) B_0 B_1 \;\le\; 2 + M_1. \tag{13}$$

Proof. $|B_0 + B_1| = 1 + B_0 B_1$ and $|B_0 - B_1| = 1 - B_0 B_1$, so $S \le \int p_0 (1 + B_0 B_1) + \int p_1 (1 - B_0 B_1)$; the second step is $|B_0 B_1| = 1$. □

This is the one-sided relaxed Bell inequality [4]. The settlement meets both steps with equality at the clamps of Eq. (6): $B_0 B_1 = \operatorname{sgn}(p_0 - p_1)$ on the whole circle and $S = 2 + M_1 = 2\sqrt{2}$. The equality holds at every separation of the host clamps. For host clamps an angle θ apart, $0 \le \theta \le \pi$,

$$M_1(\theta) = 2\big(\sin\tfrac{\theta}{2} + \cos\tfrac{\theta}{2}\big) - 2 = 2\sqrt{1 + \sin\theta} - 2, \tag{14}$$

and with A's clamps at that separation the largest S over B's clamps on the kernel $-\cos\Delta$ is $2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2} = 2\sqrt{1+\sin\theta} = 2 + M_1(\theta)$. The excess of the settlement over the local bound is the distance between the measures of the two host clamps, never more and never less. Since $M_1(\theta) \le 2\sqrt{2} - 2$,

$$S \;\le\; 2 + M_1 \;\le\; 2\sqrt{2} \qquad \text{on every clamp quartet:} \tag{15}$$

Tsirelson's value is the ceiling of the projection measure. A measure free to depend on both clamps obeys $S \le 2 + 3M$ and reaches 4 at $M = 2/3$ [3], so $2\sqrt{2}$ there needs $M \ge (2\sqrt{2} - 2)/3 = 0.276142$; the projection measure stops at $2\sqrt{2}$. $F = 2 - \sqrt{2}$ is the optimum for a measure set by one side that returns the singlet table [5].

For the host drawn per trial with weight 1/2 the table is the same (Theorem 1) and the dependence is shared between the wings: at the clamps of Eq. (6), $M_1 = M_2 = \sqrt{2} - 1 = 0.414214$, $M = 0.613126$ and $F = 0.693437$.

The mutual information between the phase and the host clamp is 0.201752 bits for two equiprobable clamps a quarter turn apart and $\log_2(\pi/e) = 0.208801$ bits for a clamp uniform on the circle.

The same weight by selection. The weight $|\cos(\varphi - a)|$ applied as a local acceptance on a uniform phase is a detection-selected model [8]: A accepts a trial with that probability and reads only its own clamp. The accepted pairs carry the projection measure and the cosine kernel at wing efficiency $2/\pi$; on all records, nulls scored 0, the same run stays under the local bound. Selection reaches the measure by dropping records. The settlement carries the same measure on every record.

Prediction. On every all-record ledger of the equal-weight pair the kernel is $-\cos\Delta$ at unit visibility and $|S| \le 2\sqrt{2}$. An all-record CHSH above $2\sqrt{2}$, or a unit-visibility kernel other than $-\cos\Delta$, falsifies the measure.

4 Implementation

4.1 Module map

The settlement readout lives in `fdt_settlement_readout.py`: the measure, the sampler, the records and the exact tables behind every settlement number of the stage (Table 1). The Born layer enters the stage in one role, as the table the settlement is compared against, and a test asserts that the readout module imports neither Born-layer module.

Module	Lines	Role in the stage
`fdt_settlement_readout.py`	567	Measures, sampler, records, exact arc tables, CHSH, price, selection, sphere, detector layer. Imports NumPy only.
`fdt_v20_projection_settlement.py`	901	Blocks [1]–[8], decision checks, report, ledger, figure, command line.
`test_v20_projection_settlement.py`	382	Thirty-five tests.
`fdt_v19_gradient_transport.py`	942	Scenario definitions, field draws, the transport run, and the Born-layer detector experiment used as the comparison column.
`fdt_field_transport.py`	386	Synthetic α_n field ensembles and batched ray and frame transport.
`fdt_geometric_transport.py`	229	The transport law of Eq. (8) and the gradient patches.
`fdt_optical_adapter.py`	53	Jones map of the transported frame; Stokes axis of a linear analyzer.
`fdt_ensemble_analysis.py`	227	Ensemble state, the γ identities, the clamp settings, the correlation tensor.
`fdt_quantum_reference.py`	162	The Born layer: two-qubit reference tables. Comparison only.
`fdt_detector_protocol.py`	224	Three-outcome local detector tables on the Born layer. Comparison only.
`fdt_detector_ledger.py`	94	Count summary of the v19 detector experiment.
`fdt_exact_core.py`	508	The Hensen axes of block [7].

Table 1: The twelve Python files of `simulation/`. The first three are stage v20; the nine below the rule are carried unchanged from the complete archive.

4.2 Exact arc integration

The port table of a setting pair is computed without sampling and without a closed form of the kernel. `_arc_table` works in the variable w that carries the measure, the host's impact phase. Both records are signs of cosines of w with centres c_A, c_B: for host A, $c_A = 0$ and $c_B = -\pi - \Delta$; for host B, $c_A = \pi + \Delta$ and $c_B = 0$. The routine

1. cuts the circle at the four sign boundaries $c_A \pm \pi/2$, $c_B \pm \pi/2$ and at the two kinks $\pm\pi/2$ of the measure;
2. sorts the six cuts, which bound six arcs $[\ell, r]$ on which both records and the sign of $\cos w$ are constant;
3. takes each arc's mass from the antiderivative, $\operatorname{sgn}(\cos m)\,(\sin r - \sin \ell)/4$ with m the arc midpoint, or $(r - \ell)/2\pi$ for the uniform control;

4. reads the two record signs at the midpoint and adds the mass to that cell of the 2×2 table.

The table is exact to rounding and the routine broadcasts over arrays of clamps and transports. `exact_port_table` returns the host-A table, the host-B table or their mixture $w\,T_A + (1-w)\,T_B$; `ensemble_port_tables` averages over histories. The kernel of Theorem 1 is then a result of the computation: block [1] compares it with $-\cos\Delta$ on 1,441 points.

4.3 Sampling the settlement

`sample_projection_phase` draws x uniform on $(-1, 1)$ and a sheet by a fair coin and returns $u = \arcsin x$ on the front sheet or $\pi - \arcsin x$ on the back; the density of u is $|\cos u|/4$. `sample_source_phase` pulls the impact phase back to the shared phase, $\varphi = u + a - 2\beta_A$ for host A and $\varphi = u + b - 2\beta_B - \pi$ for host B, with the host fixed or drawn per trial. `sign_records` applies Eq. (1). `settle_records` chains the two and returns both records, the phase and the host of every trial. No trial is rejected at any step: the number of record pairs returned equals the number of trials declared, and a decision check counts them.

4.4 Seeds and streams

A run has two seeds. The field seed (default 314159) drives the transport chain through the named streams of stage v19. The settlement seed (default 271828) drives every sampled settlement block through its own substream, `numpy.random.default_rng([seed, index])`: index 0 for block [2], 1 for block [5] (with the scenario index appended), 2 for block [6] (with the model index appended), 3 for block [4], 4 and 5 for block [8]. Changing the sample size of one block never moves another block's stream.

4.5 Detector tables and counts

The exact tables of Eq. (11) come from `exact_nine_cell_tables`, built on the exact port tables. The experiment itself runs event by event in `simulate_nine_cell_counts`: each trial draws one history, emits a pair with probability q, settles the phase on the host clamp, reads the two signs, and passes each through its own arm's detector, a click with probability $t\,\eta_{\mathrm{port}}$ and then a label flip with probability f_{port}. Every declared trial lands in one of the nine cells. `nine_cell_statistics` returns the all-trial and the coincidence-conditioned CHSH from counts or probabilities.

4.6 Quadrature, tolerances and errors

The integrals of Section 3 use the midpoint rule on 2^{20} nodes of the circle; the supremum over all clamp pairs is scanned on 721 separations of $[0, \pi]$. Checks on exact arc integration are gated at 10^{-12} and checks on quadrature at 10^{-9}. Sampled values are reported with their counting standard errors and are not gated. The standard error of a sampled CHSH is $\left[\sum_{xy}(1 - E_{xy}^2)/N_{xy}\right]^{1/2}$ with the exact correlators; the χ^2 of a nine-cell count table runs over the cells with expectation at least 5, with one degree of freedom removed per setting pair.

Measure of the shared phase	E_{00}	E_{01}	E_{10}	E_{11}	$S\pm$ SE	exact S
Projection, host A	+0.70727	+0.70711	+0.70748	−0.70688	2.828730 ± 0.001414	2.828427
Projection, host B	+0.70814	+0.70729	+0.70718	−0.70687	2.829470 ± 0.001414	2.828427
Projection, either host	+0.70589	+0.70676	+0.70789	−0.70709	2.827638 ± 0.001414	2.828427
Uniform, clamp-independent	+0.49841	+0.50082	+0.50001	−0.50095	2.000188 ± 0.001732	2.000000

Table 2: Block [2]: all-record CHSH of the sign records at the clamps of Eq. (6). Every row keeps 4,000,000 of its 4,000,000 declared records.

5 The stage, block by block

The numbers below are those of `reference/results_v20_projection_settlement.txt`, the default run: field seed 314159, settlement seed 271828, 1,536 histories, step 2 mm.

[1] The joint impact distribution (D-B1). Exact arc integration on 1,441 points of $\Delta \in [-2\pi, 2\pi]$. The largest $|E + \cos\Delta|$ is 4.4×10^{-16} for host A, 5.6×10^{-16} for host B and 3.3×10^{-16} for the host mixture; both marginals sit at 1/2 to 1.1×10^{-16}; the tables for host A and for host mixtures of weight $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ agree to 1.7×10^{-16}. The uniform control returns the triangle of Eq. (5) to 3.3×10^{-16}. At the clamps of Eq. (6), $S = 2.828427125$ for the settlement and 2.000000000 for the control, ratio 1.414213562. Over 331,776 clamp quadruples on a 15° grid and the four CHSH facets the largest $|S|$ is 2.828427125 for the settlement and 2.000000000 for the control.

[2] Every record kept. One million records per setting pair, four setting pairs, all 4,000,000 records of each row in the statistic (Table 2). Sampled single-wing frequencies sit within 0.0012 of 1/2 in every row. The sum of Eq. (7), read on each sampled phase, is +2 on every phase of every row.

[3] The clamp route priced (D-B2). On the clamps of Eq. (6): host A, $M = M_1 = 0.828427125$, $M_2 = 0$, $F = 0.585786$; host B, the mirror image; host mixture, $M = 0.613125930$, $M_1 = M_2 = 0.414213562$, $F = 0.693437$. The supremum of M_1 over all clamp pairs is 0.828427125 at clamps 90.0° apart. The one-sided bound: $S = 2.828427125$, $2 + M_1 = 2.828427125$, slack 1.2×10^{-12}, and $\int (p_0 - p_1) B_0 B_1 \, d\varphi = 0.828427125 = M_1$. Mutual information 0.201752 and 0.208801 bits.

[4] The same weight by selection. One million records per setting: accepted-pair $S = 2.828682$ at wing efficiency 0.636600 ($2/\pi = 0.636620$); $S = 1.800740$ with nulls scored 0 and 1.801084 with nulls scored −1. The settlement beside it: $S = 2.827638$ with every record kept.

[5] The transport chain on the settlement. For each of the nine scenarios the stage integrates the settlement's port table exactly on each history's clamps, averages over the 1,536 histories and compares the sixteen (setting, record) cells with the Born-layer table of the same Jones maps (Table 3). The largest cell difference over all scenarios is 7.2×10^{-15} and the largest difference between host rules 2.8×10^{-16}. With 250,000 sampled records per setting the sampled S lies within 1.66 counting standard errors of the exact value in every scenario.

Scenario	S, Born layer and settlement	cell diff.	sampled $S\pm$ SE	z	S uniform
`uniform_fluctuations`	2.828427125	7.2×10^{-15}	2.825576 ± 0.002828	-1.01	2.000000
`matched`	2.828427125	7.2×10^{-15}	2.828008 ± 0.002828	-0.15	2.000000
`independent`	1.750213217	6.7×10^{-16}	1.753728 ± 0.003596	$+0.98$	1.364052
`partial_shared`	1.882046657	2.2×10^{-16}	1.886984 ± 0.003530	$+1.40$	1.447986
`shared_unequal_baselines`	1.450123196	2.8×10^{-16}	1.449224 ± 0.003727	-0.24	1.127653
`baseline_low`	2.563461298	3.3×10^{-16}	2.562912 ± 0.003070	-0.18	1.887369
`baseline_high`	0.651282774	3.1×10^{-16}	0.655144 ± 0.003947	$+0.98$	0.529043
`ordered_regions`	2.828427125	7.2×10^{-15}	2.823728 ± 0.002828	-1.66	2.000000
`reordered_regions`	2.759170303	5.6×10^{-16}	2.756456 ± 0.002893	-0.94	1.961892

Table 3: Block [5]: the nine transport scenarios. The Born-layer and the settlement values of S agree to every printed digit; "cell diff." is the largest difference over the sixteen cells of the two tables. The last column is the same records on the clamp-independent phase.

	all-trial S		coincidence-conditioned S		
Detector model	table	sampled ± SE	table	sampled ± SE	χ^2/dof
`ideal`	1.787681	1.787400 ± 0.012535	1.750213	1.756445 ± 0.013792	7.28/16
`fixed_loss_and_readout`	1.482804	1.506600 ± 0.013008	1.660082	1.662293 ± 0.015782	33.66/32
`history_correlated_collection`	1.223112	1.228200 ± 0.012622	2.316859	2.323252 ± 0.026982	20.58/32

Table 4: Block [6]: the detector experiment on settlement records. "Table" is the exact settlement value, equal to the Born-layer value to every printed digit. The coincidence-conditioned 2.32 of the third model is selection by the collection; its all-trial value is 1.22.

[6] The detector experiment, event by event. Independent-field scenario, 85% pair source, no click scored -1, 20,000 trials per setting (Table 4). The exact settlement nine-cell tables equal the Born-layer tables to 1.2×10^{-15} over all 36 cells of each model; each is normalised to 1.3×10^{-15}; and each wing's complete three-outcome marginal is unchanged under the remote clamp to 1.3×10^{-15}.

[7] The Hensen axes, without data. At the spin axes of the event-ready experiment [10], $a=(0,\pi/2)$ and $b=(-3\pi/4-\epsilon,\,3\pi/4+\epsilon)$ with $\epsilon=0.026\pi$, the settlement gives $S=2.818997$, the value of the kernel $-\cos\Delta$ at those axes, and the uniform control gives 2.000000.

[8] The sphere. Two million records per pair at four axis pairs off one great circle, $\widehat{\mathbf{a}}\cdot\widehat{\mathbf{b}}=0$, 0.511959, 0.192450 and 0.994907: the sampled E lies within 0.63 counting standard errors of $-\widehat{\mathbf{a}}\cdot\widehat{\mathbf{b}}$ and both marginals within 0.0007 of 1/2. The great-circle section of the disc measure matches $|\cos|/4$ at Kolmogorov distance 0.000730 (1% critical value 0.001153). At the Horodecki-optimal axes of the independent-field scenario, through the per-history transport, the records give $S=2.353295\pm0.001082$ against $2\sqrt{1+|\gamma|^2}=2.352637$.

6 Outputs

A run writes three files into its output directory.

`results_v20_projection_settlement.txt`. The report: header with seeds and sample sizes, blocks [1]–[8], the reproducibility statement, the eleven decision checks, the sampling agreement and the audit line.

`settlement_ledger_v20.json`. The count ledger of block [6], schema `fdt.settlement-ledger/1`. Its counts are synthetic records of the simulation.

Key	Content
`schema`, `stage`, `data_kind`, `readout`	Identification of the ledger and of the readout that produced it
`settings_a_doubled_angle_rad`, `settings_b_doubled_angle_rad`	The clamps of Eq. (6)
`outcomes_rows_alice_columns_bob`	`["+", "-", "no_click"]`
`facet`, `no_click_assignment`	`[[1, 1], [1, -1]]` and -1
`run_parameters`	Seeds, histories, step, trials per setting, pair emission, collection width, the two fixed responses
`standard_errors`	How the two standard errors are formed
`models.<name>.settings[4]`	Per setting pair: the 3×3 integer `counts`, `trials`, `coincidences`, `settlement_probabilities` and `born_layer_probabilities`
`models.<name>.all_trial_chsh`, `.coincidence_selected_chsh`	`born_layer`, `settlement`, `sampled`, `sampled_se`
`models.<name>.chi_square`, `.degrees_of_freedom`	Agreement of the counts with the settlement table
`models.<name>.max_cell_difference_settlement_minus_born`	Agreement of the settlement table with the Born-layer table

`fig_v20_projection_settlement.png`. Figure 2, six panels.

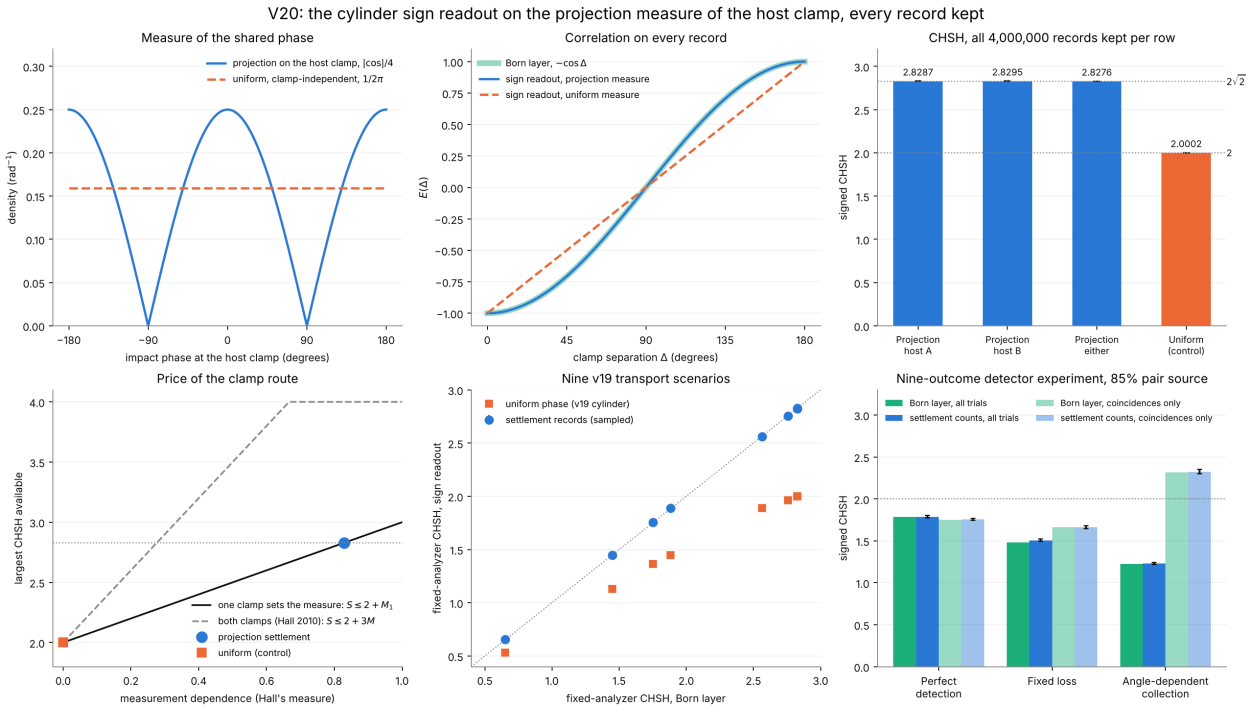


Figure 2: The stage's figure. Top row: the two measures of the shared phase; the correlation of the sign records on each, against $-\cos\Delta$; all-record CHSH with 4,000,000 records kept per bar. Bottom row: the one-sided and two-sided relaxed bounds with the settlement on the one-sided line at $M_1 = 2\sqrt{2} - 2$; the nine transport scenarios, settlement records against the Born layer, with the uniform-phase readout beside them; the detector experiment, settlement counts against the Born-layer tables.

7 Verification

7.1 Decision checks

The stage gates eleven implementation invariants and exits with nonzero status when one fails.

1. The sign readout on the projection measure returns $E = -\cos\Delta$ to 10^{-12} on the whole scan, for every host rule.
2. Both single-wing marginals are $1/2$ to 10^{-12} at every scan point and in every scenario and setting.
3. The table is the same whichever clamp hosts the settlement (10^{-12}), on the scan and in every scenario.
4. The clamp-independent measure returns the triangle, $S = 2$ at the clamps of Eq. (6), $|S| \le 2$ on the quadruple grid, and the sum of Eq. (7) lies in $\{-2, +2\}$ at every sampled phase.
5. The projection measure gives $S = 2\sqrt{2}$ at those clamps and $\max|S| = 2\sqrt{2}$ on the quadruple grid.
6. $M_1 = 2\sqrt{2} - 2$ and $S = 2 + M_1$ to 10^{-9}; the other host's dependence is zero.
7. In all nine scenarios the settlement table equals the Born-layer table cell by cell to 10^{-12}.
8. Every nine-cell settlement table is normalised, unchanged in each wing's marginal under the remote clamp, and equal to the Born-layer table to 10^{-12}.
9. Every declared trial is recorded in every sampled block.
10. At the Hensen axes the settlement returns the $-\cos$ kernel value and the uniform control returns 2.
11. The circle clamps of the settlement are the Stokes analyzer axes $(\sin a, 0, \cos a)$ of the transport chain.

7.2 Tests

The thirty-five tests fall in seven classes (Table 5). They run in `unittest` with no further dependency, and `run_tests.sh` runs them all.

7.3 Reproducibility

Under the same seeds and the same runtime the report, the ledger and the figure are reproduced byte for byte, and a test asserts it. The reference outputs were produced on Linux x86-64 with Python 3.13.18, NumPy 2.5.3, SciPy 1.18.1 and Matplotlib 3.11.2. The same package on macOS arm64 with Python 3.14.5, NumPy 2.3.5, SciPy 1.16.3 and Matplotlib 3.10.7 passes the thirty-five tests and the eleven checks and reproduces every sampled value of the report digit for digit; the largest difference anywhere in the report is 2.2×10^{-16}, in a printed residual, and the ledger agrees to 10^{-12}. The figure's bytes depend on the renderer build.

`tools/compare_to_reference.py` makes this comparison for any run: the report line by line, with numbers equal as printed or to 10^{-9}; the ledger entry by entry, counts exactly and probabilities to 10^{-12}. `MANIFEST.sha256` lists the SHA-256 of every file; `shasum -a 256 -c MANIFEST.sha256` on macOS or `sha256sum -c MANIFEST.sha256` on Linux checks them. A run writes into `output/` and leaves `reference/` and the manifest untouched.

Class	Tests	What is asserted
`MeasureTests`	4	Normalisation and CDF of the measure; the sampler uniform across the projected aperture; the pull-back puts the measure on the host clamp; invalid inputs rejected
`KernelTests`	7	The cosine kernel; marginals 1/2 and blind to the remote clamp; host and weight independence; transport through Δ alone; the triangle and the arcsine rounding; $2\sqrt{2}$, 2 and their ratio; no quartet above $2\sqrt{2}$ or 2
`RecordTests`	5	Records equal the cylinder formulas on a given phase; $2\sqrt{2}$ with every record kept; the control at the local bound; the arithmetic of Eq. (7) on every phase of either measure; seeded reproducibility
`ClampRouteTests`	5	Closed forms of Hall's measure; the one-sided bound met with equality, and holding at other quartets; mutual information; selection carries the measure and loses records
`ChainTests`	8	Ensemble and nine-cell tables equal the Born-layer tables; perfect detection reduces to the port table; extinction records every trial as no click; counts follow the exact table; matched histories return $2\sqrt{2}$; the Hensen value; the uniform measure reproduces the v19 cylinder table
`SphereTests`	4	Density and height law of the disc measure; $-\widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}}$ with flat marginals; the circle measure as its great-circle section; the transported sphere against the correlation tensor
`V20StageTests`	2	Report, ledger and figure byte-reproducible under the same seeds; the Born layer absent from the readout module

Table 5: The tests of `test_v20_projection_settlement.py`.

The first example prints the all-record S of both measures at one million records per setting. The second evaluates Eq. (14) and the bound at 181 separations from 0 to 180°: the quadrature agrees with the closed form to 1.7×10^{-10} and the largest S over B's clamps equals $2 + M_1(\theta)$ to 8.9×10^{-16}.

8 Using the readout as a library

`fdt_settlement_readout.py` depends on NumPy alone and is usable without the rest of the package.

```
import numpy as np
import fdt_settlement_readout as sr

rng = np.random.default_rng(1)

# one million record pairs at clamps (a, b), host drawn per trial
A, B, phi, host_is_a = sr.settle_records(rng, 1_000_000, a=0.0, b=-3*np.pi/4)
print((A * B).mean(), A.size)      # 0.7059 on 1000000 records; exact 0.7071 = -cos(3 pi/4)

# the exact table of the same setting pair, and its correlation
table = sr.exact_port_table(0.0, -3*np.pi/4)          # 2 x 2, rows A, columns B
print(sr.correlation(table), sr.marginals(table))     # 0.70710678..., (0.5, 0.5)

# the clamp-independent control: same records, uniform phase
table = sr.exact_port_table(0.0, -3*np.pi/4, measure="uniform")
```

```
print(sr.correlation(table))                          # 0.5, the triangle

# CHSH at the default clamps, and the price
print(sr.chsh_value(sr.chsh_tables()))                # 2.8284271...
print(sr.one_sided_bound())                           # S, M_1, bound, slack
```

Function	Returns
`projection_density`, `uniform_density`, `projection_cdf`	The two measures and the CDF of the first
`sample_projection_phase`, `sample_source_phase`	Impact phases at the host clamp; shared phases and the host of each trial
`sign_records`, `settle_records`, `counts_2x2`	The records of Eq. (1); sampled records with their phases; a 2×2 count table
`relative_angle`, `jones_angles`	Δ; β of real proper Jones maps
`exact_port_table`, `ensemble_port_tables`	Exact tables for clamps and transports; history averages for a CHSH quartet
`correlation`, `marginals`, `chsh_tables`, `chsh_value`	E; $P(A{=}+)$, $P(B{=}+)$; the four tables of a quartet; S
`counterfactual_chsh`	The sum of Eq. (7) on each phase
`host_density`, `measurement_dependence`, `continuous_measurement_dependence`	$p(\varphi \mid a, b)$ for a host rule; M, M_1, M_2, F; the supremum over clamp pairs
`one_sided_bound`, `mutual_information_bits`	S, M_1, the bound and its slack; the two information values
`selection_exhibit`	The detection-selected route beside the settlement
`sample_disc_measure`, `sphere_records`, `sphere_settlement`, `circle_section_statistic`	The disc measure, its records, and the Kolmogorov distance of its great-circle section
`response_matrix`, `exact_nine_cell_tables`, `simulate_nine_cell_counts`, `nine_cell_statistics`	The detector layer of Eqs. (10)–(11)

9 Command line

Option	Default	Meaning
`--out-dir`	`simulation/`	Output directory; `run.sh` passes `./output`
`--seed`	314159	Field draws of the transport chain
`--settlement-seed`	271828	Every sampled settlement block
`--histories`	1536	Histories per scenario (at least 8)
`--step-m`	0.002	Transport step in metres, in $(0, 0.01]$
`--trials`	1,000,000	Records per setting, blocks [2] and [4]
`--scenario-trials`	250,000	Records per setting, block [5]
`--detector-trials`	20,000	Trials per setting, block [6]
`--sphere-trials`	2,000,000	Records per axis pair, block [8]
`--quick`	off	Caps: 256 histories; 200,000, 50,000 and 400,000 records in blocks [2] and [4], [5], [8]
`--no-plots`	off	Skip the figure

Each sample size is at least 1,000. The exact blocks [1], [3] and [7] do not depend on any sample size.

10 Status and scope

Item	Status	Where it binds
The settlement measure $p_{\mathrm{host}}(\varphi \mid a) = \lvert\cos(\varphi - a)\rvert/4$ on the host clamp	stated by the core (I-4, v6.7)	every settlement number of the stage
The joint impact distribution on the cylinder readout, its cosine kernel, flat marginals, host invariance and nine-cell detector tables	derived from that measure (D-B1)	blocks [1], [2], [5]–[8]
The parameter and prediction of the clamp route: $M_1 = 2\sqrt{2} - 2$, $S \le 2 + M_1$ met with equality, ceiling $2\sqrt{2}$	computed (D-B2)	block [3]
The dynamics of the settlement — what selects the host clamp, how the measure is carried between the clamps — and the measure for a pair of unequal weights or for more than two patterns	to be derived (O-17)	no number of the stage uses it
$n_{\mathrm{cone}} = (1 - \alpha_n)^{-1/2}$ as the aperture assignment	the core's assignment (P-4)	the transport chain of blocks [5], [6], [8]
The constitutive derivation of n_{cone}	to be derived (O-8)	the same
The α_n application of the bounded map	a working token of the simulation	the field law of the transport chain

The measure is written for the equal-weight pair: the singlet and every transported or history-averaged image of it. The host rule does not enter the table, which is the same for host A, host B and every mixture (Theorem 1); what selects the host is part of O-17. The fields, histories and counts of the stage are synthetic and seeded. The loophole-free experimental record is accepted in full. The event-ready experiment of Hensen et al. [10] measures the equal-weight pair, and the stage evaluates its axes in block [7].

11 Convergence statement

On the same source and the same settings the stage recovers the consensus table of the equal-weight pair: $-\cos\Delta$ with flat marginals on every record, the ensemble tables of the nine transport scenarios and the nine-outcome detector tables cell by cell, with no free number and with $2\sqrt{2}$ as the ceiling. FDT claims no divergence from that table. FDT declines the consensus readings of it — nonlocal action, collapse, probability amplitudes, a hidden state independent of the clamps — and reads the same record as shared bivector grade with deterministic phase, the pair's record one settlement of the pair and its clamps (I-4, 0.1a). The premise of Bell's theorem that the settlement declines is measurement independence, and its price is $M_1 = 2\sqrt{2} - 2$, the whole of the distance from the local bound 2 to Tsirelson's $2\sqrt{2}$.

Author contributions

M. A. developed Fundamental Density Theory, its locked core and the simulation archive through stage v19.1, posed the problem, directed the work and adopted the measure into the core as the pair's record of I-4 (v6.7). Claude Fable 5.2 found the projection measure of the host clamp, proved

Theorem 1 and Proposition 1, wrote and tested stage v20, and wrote this note. M. A. is the corresponding author.

Distribution

The software of this package is released under the MIT License (`LICENSE`). To cite the work, cite the companion paper [1]. The complete simulation archive, with every earlier stage, is available from the author on request. The project's site is `https://densitytheory.com`.

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