

What ψ Is

One density cone at every size: Kepler's conic sections, helical orbits,
and quantum mechanics in cylindrical form

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Fundamental Density Theory

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Abstract

In Fundamental Density Theory the density coordinate α draws a truncated cone. Its circles are the concentric circles of the Alpha Lock, $\alpha_{\text{ext}} = R_s/z$; its generators stand at $\pi/4$ in the far field, where the density cone is the flat light cone of uniform density (P-4); and it is truncated at the fold, a circle and never a point. Place the Sun at the tip. The light cone of the Sun meets the world-tube of every orbit, the cylinder the orbit sweeps out in time, in a plane. The section is a conic, the slope of the plane against the cone's generators is the eccentricity, and the plane parallel to a generator carries the orbit that moves at the core's flow speed $c\sqrt{\alpha_{\text{ext}}}$ at every radius. Pressed flat onto its base, the section is Kepler's conic, with the Sun at the focus, where the cone's axis meets the base, inside the projected fold circle. The pair law gives the second law, and the cone's own coordinate gives the third, $(v/c)^2 = \alpha_{\text{ext}}(r) - \alpha_{\text{ext}}(a)/2$. The field equations carry the next order in α_{ext} and turn the section by $3\pi\alpha_{\text{ext}}(\ell)$ per revolution, $42.98''$ per century for Mercury. A body's worldline winds on its world-tube as a helix. Projected along the system's own time axis it is the closed orbit; projected along the time axis of an observer in whose frame the Sun moves along the Galactic rotation at 248.5 km s^{-1} , it is a helix in space, advancing 52.4 AU per turn for the Earth: a giant body moving in a wave. Shrinking the cone changes no angle and no ratio of the orbit's own lengths. It changes one number, the action of an orbit in units of $\hbar$, the action that fixes the floor: 2.5×10^{74} for the Earth, a few units for an atom. In the cylindrical coordinates of the helix the Schrödinger equation is two real equations, continuity for the helix's radius and Hamilton–Jacobi for its phase angle with a profile term, together with the closure of the helix on its cylinder. The unit imaginary is the oriented unit area of the plane normal to the axis, and $\psi = \rho e^{i\varphi}$ is the projection of the helix onto that plane: a point turning in time for an observer inside the system, the de Broglie wave for a bookkeeper outside it. In a circular state of principal number n the profile term carries $1/n$ of the kinetic energy: 4×10^{-75} for the Earth's orbit, 1.4% at $n = 72$, the principal number of the electron that Maeda and Gallagher held on a Kepler orbit, and all of it in the ground state of hydrogen. On the cylinder the quantum phenomena read as its geometry. The de Broglie wave is the outside observer's reading of a pattern's own circulation, quantization is closure, interference is the law of cosines, spectral lines are beats, the commutator is an oriented area, tunnelling is a wall made porous by the omnium's motion, the zero-point energy is the profile term, spin is a rotor on the three-sphere of two ports, and entanglement is one pattern with a shared bivector grade, read at two probes. A receipt script checks the displayed identities and numbers: 96/96 pass.

1 The cone, the orbit and the helix

Kepler found the planets on ellipses with the Sun at a focus, sweeping equal areas in equal times, with the squares of their periods in the ratio of the cubes of their mean distances [5, 6]. Three centuries later Rutherford found alpha rays deflected by the atomic nucleus along hyperbolas, Kepler's open orbits, with the nucleus at the focus [7, 8]. Bohr and Sommerfeld put the electron on Kepler's circles and ellipses and closed them in units of $\hbar$ [9, 10]. Pauli derived the hydrogen spectrum from the same conserved vector that fixes the orientation of a planetary ellipse [11]. Radial electron wave packets have been timed at Kepler's periods [21], and in 2004 one was held on a Kepler orbit for 15 000 revolutions [22]. The orbits of the Solar System and of the atom are one geometry. This paper reads that geometry through the locked core v6.2 [1], with the companion derivations of c and $F_{\max}$ [3, 4].

The density coordinate α draws a truncated cone, and the cone is the light cone (P-4). Place a source at its tip, and every orbit about it is a plane section of the cone: the light cone of the source meets the world-tube of the orbit, the cylinder the orbit sweeps out in time, in a plane (§3). Press the cone flat onto its base and the section is Kepler's conic with the source at the focus. The cone's own coordinate gives the third law, and the field equations carry what pressing flat leaves out (§4). The orbit's worldline winds on the world-tube as a helix. The two readings of the core decide what is seen. Projected along the system's own time axis, the worldline is the closed orbit; projected along the time axis of an observer in whose frame the source moves, it is a helix in space. The Earth runs such a helix about the Sun's track through the Galaxy, a giant body moving in a wave (§5). Shrink the cone to atomic size and nothing in the geometry changes but the action per turn in units of $\hbar$, the action of the floor (§6). On the cylinder of the helix the complex equations of quantum mechanics are real cylindrical equations, and every quantum phenomenon reads as geometry of the helix (§§7–8).

The organizing question is the observer's place. An observer inside the system reads the closed orbit, and the phase of a pattern as a point turning in a plane, the complex plane of quantum mechanics. An observer outside the system, for whom the system moves, reads the same worldline as a helix and the same phase as a wave. The core names the two the pattern's reading and the bookkeeper's reading (0.1c). The receipt script of Appendix A checks the displayed identities and numbers; check labels appear in brackets, [H07a].

2 Premises, charts and symbols

The derivation uses the locked core v6.2 [1] and the measured record. Each premise is quoted by section.

- (P1) *One substance, only patterns.* There is one substance, the omnium, and there are only its patterns; every physical length and time belongs to a pattern, to a pattern-to-probe measurement, or to the floor (0.1, 0.1b).
- (P2) *The chain, the strand rate, the floor.* $E/m = c^2 = 1/(\epsilon_0\mu_0) = (d/t)^2$ is one identity (0.1, 0.1a); c is the strand rate of the omnium (0.1d; [3]); $F_{\max} d_{\min} t_{\min} = \hbar$ fixes the floor $d_{\min} = 2\ell_P$, $t_{\min} = 2t_P$, the infimum of physical strokes, approached by the least closed pattern and occupied by none (0.1a; [4]). [H31]
- (P3) *The Alpha Lock.* On the exterior branch $\alpha_{\text{ext}}(r_f) = R_s/r_f = \kappa(r_f)/\kappa(R_s)$, the ratio of the curvatures of two concentric circles, with $R_s = 2GM/c^2 = \mathcal{M}c^2/(2F_{\max})$; $0 < \alpha_{\text{ext}} < 1$,

the ends limits and never states (0.2, 0.3, 0.4, I-6). On the A-1 compactness chart, exterior branch, $\alpha_{\text{ext}} < 1/2$ is the general-relativistic regime, the far reading (0.3a).

- (P4) *Density cones are light cones.* The truncated density cone at a point is the local light cone (P-4). A source sits at the tip of its cone, where density and energy are greatest; the wide rim is the far field (I-3).
- (P5) *The pair law and the field equations.* $F = F_{\text{max}}\alpha_1\alpha_2 = G\mathcal{M}_1\mathcal{M}_2/d^2$, exactly, on two separate spherical supports (I-1, I-4 layer L1). $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ stands unchanged, and for the same source and data FDT returns the curvature of general relativity (I-3, containment; I-4 layer L2).
- (P6) *Two readings.* A pattern's own clocks and rulers and a distant bookkeeper's read one geometry; the uniform motions that carry strokes to strokes form the boost group (0.1c, 0.1d).
- (P7) *The quantum face and closed patterns.* Quantum mechanics is helical geodesics on truncated density cones: the wavefunction $\psi = Ae^{i\varphi}$ is the projection of a helix onto the plane normal to the cone axis, superposition is ignorance of the helical phase φ_0 at observation, and the universe is deterministic (I-4). Matter is closed stroke; mass is a confinement eigenvalue, and its operator is to be derived (I-5; D-02, core O-9). A closed pattern's phase has a phase rate ω/k and a run $d\omega/dk$ whose product is c^2 (I-2).
- (P8) *Settlement.* Every step is a system-wide settlement of the state it starts from; responses are many-to-many, never particle by particle (0.1a).

Charts. One density coordinate appears: $\alpha_{\text{ext}} = R_s/r_f$, application A-1 (compactness), exterior branch $\sigma = +1$, for the Sun as source, with $R_s = 2G\mathcal{M}_{\odot}/c^2 = 2953.25$ m from the measured $G\mathcal{M}_{\odot}$, every energy of the Sun entering $\mathcal{M}_{\odot}$ once (I-7). Its field radius r_f on the cone is the height z of a circle, and the radius r of (6) and (8) is this r_f . At the field radii of §§3–5 it takes values from 4.245×10^{-6} at the solar surface down: the field radii of the planetary orbits lie in the general-relativistic regime, the far reading of 0.3a. [H08a, H08b] In the pair law (3), α_1 and α_2 are the source's and the body's α_{ext} at their separation r , each on its own exterior branch (I-1). At the atomic scale the same plane sections hold for every inverse-square pair, but the map from a charge to an I-1 input is to be derived (D-01; core O-6), so no α is assigned to an atomic orbit. The shares of quantum mechanics are port shares: they reach both ends of their range and are never an α (I-6), though the state space of two ports in §8 is a three-sphere of the same form as the support of 0.3a. The helix radius $\rho = |\psi|$ is the square root of a share density, not a length: the angle the ψ -helix makes with its axis depends on the scale chosen for ρ and carries no datum, while its advance per turn and its two rates do; the stroke's helix of I-2, of radius $\bar{\lambda} = c/\omega$, stands at $\pi/4$.

symbol	meaning
z	height on the cone, $z = ct$; on the cone $z = r$, the distance from the source
$\mathcal{M}, \mathcal{M}_\odot$	total gravitating energy of a source, in mass units (0.5); the Sun's
$m, m_\oplus$	mass of a body or pattern; the Earth's
m_e	the electron's confinement-position mass at helix count $n = 3$, a CODATA input (D-02)
K	constant of an inverse-square pair, $F = K/r^2$: $G\mathcal{M}m$ for gravity, $q_e^2/(4\pi\epsilon_0)$ for hydrogen
$\mathbf{L}, \mathbf{A}$	angular momentum; Laplace–Runge–Lenz vector $\mathbf{p} \times \mathbf{L} - mK\hat{\mathbf{r}}$
E	energy of an orbit or of a state
e, ℓ, a	eccentricity $ \mathbf{A} /(mK)$; semi-latus rectum $L^2/(mK)$; semi-major axis
T, ϖ	period; longitude of perihelion
$V, \mathbf{V}$	speed and velocity of a source in an outside observer's frame
U	potential energy of a pair (the U of I-1)
$\psi = \rho e^{i\varphi}$	the projection of a helix point (z, ρ, φ) onto the plane normal to the axis
$\mathbf{e}_1, \mathbf{e}_2, \boldsymbol{\Sigma}$	orthonormal vectors of that plane; its unit bivector $\boldsymbol{\Sigma} = \mathbf{e}_1\mathbf{e}_2$, $\boldsymbol{\Sigma}^2 = -1$
Q_ρ	profile term $-(\hbar^2/2m)\nabla^2\rho/\rho$
$\varkappa, J$	decay rate of the radius in a barrier; transmitted current
Θ, χ	Rutherford scattering angle; Stern–Gerlach angle
$C(\mathbf{a}, \mathbf{b})$	correlation of a two-probe state along analyser axes $\mathbf{a}, \mathbf{b}$
ω_0	rest circulation of a closed pattern, $mc^2/\hbar$
n, l, m_l, j	principal number (and winding); orbital and magnetic numbers; SO(4) label
q_e, a_0, v_1, T_1	elementary charge; Bohr radius $\hbar^2/(m_e K)$; $K/\hbar$; $2\pi a_0/v_1$
e	the exponential base, set upright to keep it apart from the eccentricity e

3 The density cone and its plane sections

3.1 α draws the cone

Take the source's own frame, with its worldline as the axis, and draw the plane of an orbit with the time $z = ct$ above it. The light cone of the source, with the source's worldline as its axis, is

$$\mathcal{K} = \{(x, y, z) : z = r > R_s\}, \quad r = \sqrt{x^2 + y^2}, \quad (1)$$

and its circle at height z has radius z and curvature $1/z$. The Alpha Lock reads the cone directly:

$$\alpha_{\text{ext}}(z) = \frac{\kappa(z)}{\kappa(R_s)} = \frac{R_s}{z}, \quad (2)$$

the ratio of the curvature of the circle at height z to that of the circle at height R_s (0.3). [H01a] Under $(R_s, z) \mapsto (\lambda R_s, \lambda z)$ every α is unchanged, and so is the cone: it is the self-similar surface that α 's circles sweep, one form at every size (0.3, 0.3a). [H01b] Its generators are the lines $z = r$, the stroke lines $x = ct$, standing at $\pi/4$ in the plane (x, ct) , the angle at which every stroke's helix stands (I-2). The cone is truncated at the fold, $z = R_s$, where $\alpha_{\text{ext}} \rightarrow 1^-$: a circle, never a point and never reached (0.3a, I-6). Density rises toward the tip and falls toward the rim (I-3), and the source sits at the tip. This is the cone of the far field, where the planets are; the aperture of the density cone near the fold is to be derived (D-05; core O-8) and does not enter.

The source acts on a body of mass m at distance r through the pair law. On two separate spherical supports at monopole order

$$F = F_{\text{max}} \alpha_1 \alpha_2 = \frac{c^4}{4G} \frac{2G\mathcal{M}}{c^2 r} \frac{2Gm}{c^2 r} = \frac{G\mathcal{M}m}{r^2}, \quad (3)$$

exactly in model (I-1, I-4 layer L1). [H01c] It is an inverse-square pair, $F = K/r^2$ with $K = GMm$. The geometry below needs only that form. The electron and the proton of an atom, the same $n = 3$ braid read from the free room and from the nuclear room (I-5), form such a pair with $K = q_e^2/(4\pi\epsilon_0)$, read through I-1's "one equation, four names", whose map from a charge to an I-1 input is to be derived (D-01; core O-6).

3.2 The cone and the cylinder meet in a plane

Theorem 1 (The plane section). *Let a body move under an inverse-square attraction $F = K/r^2$ toward a source at the origin, with angular momentum $\mathbf{L}$ and Laplace–Runge–Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mK\hat{\mathbf{r}}$, both conserved. Lift every point of its orbit to the light cone, $z = r$. The lifted orbit lies in the plane*

$$z = \ell - \mathbf{e} \cdot \mathbf{x}, \quad \ell = \frac{L^2}{mK}, \quad \mathbf{e} = \frac{\mathbf{A}}{mK}, \quad e = |\mathbf{e}|, \quad (4)$$

so the light cone of the source meets the world-tube of the orbit in a plane section. Conversely, every plane $z = \ell - \mathbf{e} \cdot \mathbf{x}$ with $\ell > (1 + e)R_s$ meets the cone in the lift of an orbit, its perihelion $\ell/(1 + e)$ outside the fold. The height of the plane on the axis is fixed by L and its tilt by $\mathbf{A}$, and

$$e^2 = 1 + \frac{2EL^2}{mK^2}. \quad (5)$$

Proof. Since $(\mathbf{p} \times \mathbf{L}) \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{L} = L^2$, $\mathbf{A} \cdot \mathbf{r} = L^2 - mKr$. [H02c] Dividing by mK gives $r = \ell - \mathbf{e} \cdot \mathbf{x}$, which with $z = r$ is (4); with θ measured from $\mathbf{A}$ it is the focal conic $r = \ell/(1 + e \cos \theta)$. [H02a] $\mathbf{A}$ is perpendicular to $\mathbf{L}$, so the tilt lies in the orbital plane. [H02e] The world-tube of the orbit is the cylinder $\{(\mathbf{x}, z) : \mathbf{x} \text{ on the orbit}\}$, and its intersection with $z = r$ is the lifted orbit. Conversely, a point of the cone on the plane has $r = \ell - \mathbf{e} \cdot \mathbf{x}$, the same conic, whose least radius $\ell/(1 + e)$ exceeds R_s . Finally $|\mathbf{A}|^2 = m^2 K^2 + 2mEL^2$ gives (5). [H02d] $\square$

An integrated orbit confirms it without the closed form: the lift (x, y, r) of a numerically integrated ellipse, parabola and hyperbola lies on one plane to 10^{-11} . [H03a–c] The construction holds on every right circular cone $z = kr$, with the plane $z = k(\ell - \mathbf{e} \cdot \mathbf{x})$; the light cone fixes $k = 1$, so that the slope of the plane is e itself. The section is a set of events on the cone; the body's own worldline is the helix of §5, which meets it in a single event.

Corollary 1 (The slope is the eccentricity). *The plane (4) rises with slope e against $-\mathbf{e}$; the cone's generators rise with slope 1. If $e < 1$ the plane is flatter than the light cone and cuts a closed section, an ellipse, a circle at $e = 0$, and $E < 0$. If $e = 1$ the plane is parallel to a generator, at $\pi/4$, and cuts a parabola, $E = 0$, on which $v^2 = 2K/(mr)$ at every radius. For gravity, $K/m = GM = c^2 R_s/2$ (0.2), and the parabolic orbit moves at*

$$\left(\frac{v}{c}\right)^2 = \frac{R_s}{r} = \alpha_{\text{ext}}(r), \quad (6)$$

the speed of the flow of $0.1c$, the speed of a pattern falling from rest far away. If $e > 1$ the plane is steeper than the light cone, cuts an open section, and $E > 0$. At perihelion the speed over the flow speed is $\sqrt{(1 + e)/2}$: below 1 on every closed section, 1 on the parabola, above 1 on every open one. [H05a–d]

Corollary 2 (The other nappe). *The same plane meets the other nappe, $z = -r < -R_s$, in $r = \ell/(e \cos \theta - 1)$, the orbit of a repulsive pair with the same ℓ and e . [H02b] For $e > 1$ the two branches of one hyperbola are the attractive and the repulsive orbit about one focus. On either branch*

(a)

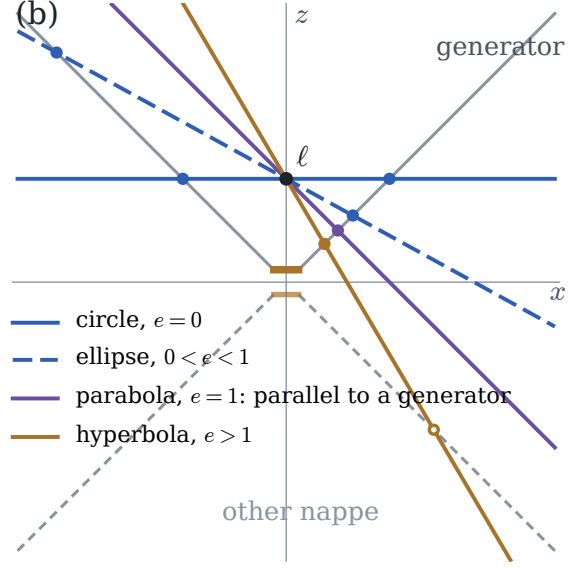
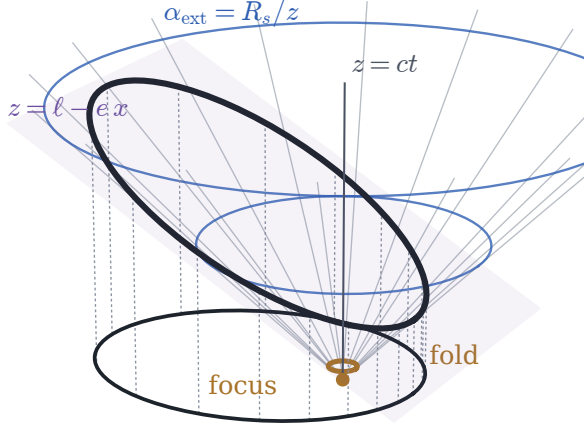


Figure 1: The cone and the cylinder meet in a plane. (a) The light cone $z = ct = r$ of the source, its circles labelled by $\alpha_{\text{ext}} = R_s/z$ and truncated at the fold (the small circle at the tip), with the world-tube of an orbit (dotted) and the plane $z = \ell - ex$ (shaded). The section (heavy) lies on the cone, the cylinder and the plane at once; pressed flat along the axis it is the focal conic below, with the focus where the axis meets the base, inside the fold circle pressed flat. (b) Edge-on, along the minor axis: every plane passes through the height ℓ on the axis and rises with slope e ; the generators rise with slope 1. Flatter than a generator, the section closes (circle, ellipse); parallel to one, it is the parabola; steeper, it opens (hyperbola), and the same plane cuts the other nappe in the repulsive branch (open circle). Both nappes stop at the fold (short bars). Drawn for $\ell = 1$; the ellipse has $e = 0.55$, the hyperbola $e = 1.7$.

the deflection obeys $\sin(\delta/2) = 1/e$, and for a repulsive pair of energy E the impact parameter is $b = (K/2E) \cot(\Theta/2)$, which gives Rutherford's cross section,

$$\frac{d\sigma}{d\Omega} = \left(\frac{K}{4E}\right)^2 \frac{1}{\sin^4(\Theta/2)}. \quad (7)$$

[H06a–c]

Rutherford derived (7) from these hyperbolas in 1911, and Geiger and Marsden confirmed it [7, 8]; the exact quantum-mechanical cross section for the Coulomb pair is the same function [14, 15]. The sign of a pair, attraction or repulsion, is set by the flow-direction rule of P-4.

4 Pressed flat: Kepler's laws

Proposition 1 (Kepler's three laws, read on the cone). (I) Projected along the axis onto its base, the section of Theorem 1 is the conic $r = \ell/(1 + e \cos \theta)$, and the axis projects onto its focus, the fold circle at the tip onto the circle of radius R_s about it: the source sits at the focus because it sits on the axis, at the tip. (II) The pair force is central, $\mathbf{L}$ is conserved, and the radius sweeps area at

the constant rate $L/(2m)$. (III) The speed at distance r on an ellipse of semi-major axis a is

$$\left(\frac{v}{c}\right)^2 = \alpha_{\text{ext}}(r) - \frac{1}{2} \alpha_{\text{ext}}(a), \quad (8)$$

so on a circle $(v/c)^2 = \alpha_{\text{ext}}(a)/2$, the period is $T = 2\pi a / (c\sqrt{\alpha_{\text{ext}}(a)/2})$, and

$$T^2 = \frac{8\pi^2 a^3}{c^2 R_s} = \frac{4\pi^2 a^3}{G\mathcal{M}}, \quad \frac{E}{mc^2} = -\frac{\alpha_{\text{ext}}(a)}{4}. \quad (9)$$

Proof. (I) is (4) with z dropped. (II): $r^2\dot{\theta} = L/m$; the integrated orbit holds dA/dt constant to 2×10^{-12} . [H04] (III): energy conservation with $U = -K/r$ and $E = -K/(2a)$ gives $v^2 = (K/m)(2/r - 1/a)$, and $K/m = c^2 R_s/2$. [H07a–c] On an ellipse, (II) gives $T = \pi ab(2m/L)$ with $b = a\sqrt{1 - e^2}$ and $L^2 = mKa(1 - e^2)$, the same T . [H07d] $\square$

At the Earth's distance $\alpha_{\text{ext}}(1 \text{ AU}) = 1.974 \times 10^{-8}$, and (8) on a circle gives $v = c\sqrt{\alpha_{\text{ext}}/2} = 29.78 \text{ km s}^{-1}$ and $T = 365.26 \text{ d}$. [H08b–d] The three laws hold exactly in model on I-1's domain (layer L1) for the relative orbit, with $\mathcal{M} + m$ in place of $\mathcal{M}$ in (8)–(9); for the Earth this shortens T by 1.5×10^{-6} of itself and leaves 365.26 d. [H08e] Pressing the cone flat keeps the leading order in α_{ext} ; the field equations carry the rest.

Proposition 2 (The field equations turn the section). *For the same source and data the field equations of I-3 turn the section by*

$$\Delta\varpi = \frac{3\pi R_s}{\ell} = 3\pi \alpha_{\text{ext}}(\ell) \quad \text{per revolution} \quad (10)$$

at leading order (I-3 containment, layer L2). For Mercury, $\ell = a(1 - e^2) = 5.546 \times 10^{10} \text{ m}$, this is $5.019 \times 10^{-7} \text{ rad per revolution}$ and $42.98'' \text{ per century}$, the anomalous advance *Le Verrier* found in 1859 and *Einstein* derived [16, 17]. [H09a]

The relativistic orbit equation, integrated for $R_s/\ell = 10^{-4}$, advances by $3\pi R_s/\ell$ per revolution to its leading order. [H09b] The flattened cone closes its ellipses, and the field equations turn them into a rosette (I-3).

Remark (A stroke is not a section). The flattened cone carries only the lapse, the slow pattern's reading (0.1c); its hyperbola of impact parameter b , taken to speed c , bends by $2G\mathcal{M}/(bc^2) = R_s/b$, $0.88''$ at the solar limb. The stroke is bent by twice that, $2R_s/b = 1.75''$ (I-3), because it reads the optical metric rather than the lapse: the factor of two of I-3 is the extended equivalence principle read on a stroke (0.1c). [H10]

5 Add the source's motion: the helix

5.1 One worldline, two readings

In the source's own frame the body's worldline $(\mathbf{x}(t), ct)$ lies on the world-tube of its orbit and winds once around the source's worldline for every cT of advance along the axis: it is a helix on the world-tube, a circular helix of radius a and pitch cT for a circular orbit. The Earth's helix in the Sun's frame has radius 1 AU and pitch one light-year, standing $\arctan(v/c) = \arctan(9.94 \times 10^{-5})$ from its axis. [H11e]

Theorem 2 (One worldline, two readings). (i) *Projected along the system's own time axis, the worldline is the closed orbit: the reading of an observer inside the system.* (ii) *Let an observer outside the system see the source move with velocity $\mathbf{V}$ along an axis normal to a circular orbit of radius a and period T . In that observer's frame the same worldline, projected along that observer's time axis, is a helix about the source's track, of radius a , period γT and advance γVT per turn, $\gamma = (1 - V^2/c^2)^{-1/2}$; seen side-on it is a wave of wavelength γVT and amplitude a . If $\mathbf{V}$ is oblique to the orbital plane the helix lies on an oblique cylinder.* (iii) *The two are one worldline read along two time axes, the system's own reading and the bookkeeper's (0.1c), related by a boost (0.1d). On the system's own clocks and rulers nothing changes.*

Proof. (i) The projection drops the time coordinate of each event and leaves the orbit $\mathbf{x}(t)$. (ii) In the source's frame the body is at $a(\cos \omega t, \sin \omega t, 0)$ at time t , with $\omega = 2\pi/T$. The boost along the axis gives $z' = \gamma V t$ and $t' = \gamma t$ and leaves the transverse coordinates unchanged. So $z' = V t'$, and the body turns once per $\Delta t' = \gamma T$ while the source advances γVT . [H11f] (iii) Both are projections of the one set of events. $\square$

5.2 The Earth on the Sun's track

The Sun moves along the Galactic rotation at 248.5 km s^{-1} , from the apparent proper motion of Sgr A*, $6.411 \text{ mas yr}^{-1}$ along the Galactic plane, at the distance $R_0 = 8.178 \text{ kpc}$ [18, 19]. [H11a] Taken as the track, that motion points toward $(l, b) = (90^\circ, 0^\circ)$, 30.4° from the pole of the ecliptic at $(l, b) = (96.38^\circ, 29.81^\circ)$. [H11b] In the Galaxy's frame the Earth's path is therefore a helix on an oblique cylinder about the Sun's track. It advances $VT = 52.4 \text{ AU}$ per turn against an orbital radius of 1 AU and stands between 5.6° and 6.9° from its axis through the year. [H11c, H11d] The Sun's further 11 km s^{-1} toward the Galactic centre and 7 km s^{-1} toward the north Galactic pole [20] turn the track to $(l, b) = (87.4^\circ, 1.7^\circ)$, 29.4° from the pole, at 248.9 km s^{-1} ; the advance becomes 52.5 AU per turn, and the range stays 5.6° to 6.9° . [H11g] Seen from outside the Solar System, the Earth is a giant body moving in a wave, and its wave is the projection of the same worldline whose projection along the Sun's time is Kepler's ellipse.

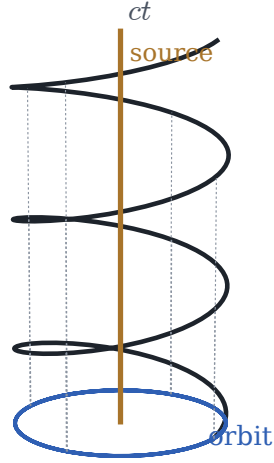
The construction nests. The Moon's path is a helix about the Earth's, the Earth's about the Sun's, and the Sun's path is a helix about the Galaxy's: helices within helices, one construction read at successive sizes (0.3a, 0.8), bounded below by the floor (0.1a).

6 Shrink the cone: the atom

6.1 The same sections at the atomic scale

Theorem 1 needs only an inverse-square pair, so the cone's sections reappear when it is shrunk to atomic size. Rutherford's alpha rays ran its hyperbolas about the nucleus [7]. Bohr and Sommerfeld put the electron on its circles and ellipses [9, 10]. Kepler's third law holds at the atom: on the orbit of principal number n , $a = n^2 a_0$, $v = v_1/n$ and $T = n^3 T_1$, with $a_0 = \hbar^2/(m_e K) = 5.29177 \times 10^{-11} \text{ m}$, $v_1 = K/\hbar = 2187.69 \text{ km s}^{-1}$ and $T_1 = 2\pi a_0/v_1 = 1.5198 \times 10^{-16} \text{ s}$. [H13a, H13c] Radial electron wave packets oscillate with these periods [21]. Maeda and Gallagher phase-locked the orbit of a lithium Rydberg electron to a 17.5 GHz field for 900 ns , roughly $15\,000$ revolutions [22]: Kepler's third law at the atom makes that orbit $n = 72$, with semi-major axis $a = 274.3 \text{ nm}$ and circular speed $v_1/n = 30.38 \text{ km s}^{-1}$, beside the Earth's 29.78 km s^{-1} . [H13d, H13e] Their packet, built from np states in a linearly polarized field, runs a highly eccentric, nearly linear Kepler orbit at the period $n^3 T_1$.

(a) inside the system



(b) outside the system

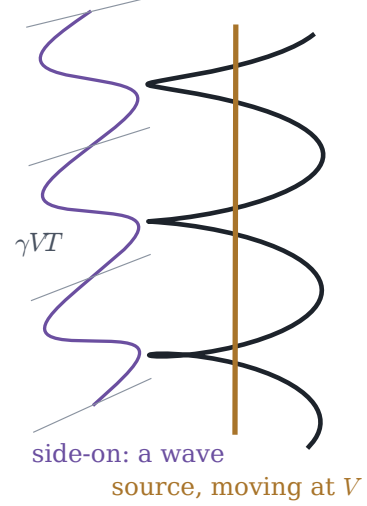


Figure 2: One worldline, two readings. (a) In the system's own spacetime the body's worldline winds on the world-tube of its orbit about the source's worldline; projected along that time axis it is the closed orbit. (b) For an observer in whose frame the source moves at V , the same worldline is a helix in space about the source's track, advancing γVT per turn; side-on it is a wave. For the Earth: radius 1 AU, a pitch of one light-year in (a), and 52.4 AU per turn in (b) in the Galaxy's frame. The drawings are not to scale.

Atoms move, so their electrons run Theorem 2 as well. Seen from outside an atom that moves at V , an electron on a Kepler orbit runs a helix about the nucleus's track, advancing γVT per turn.

6.2 What shrinking changes

The sections are scale-free, and so is α_{ext} on the cone of every source with a declared R_s . Shrinking changes one number: the action of an orbit in units of $\hbar$, the action that fixes the floor, $F_{\text{max}} d_{\text{min}} t_{\text{min}} = \hbar$ (0.1a). For one pattern on a circular orbit, $L/\hbar$ is the number of turns of its phase per revolution (§7). For the Earth, a body of many patterns [4], it is the action of the orbit per turn in units of $h = 2\pi\hbar$:

$$\frac{L_{\oplus}}{\hbar} = \frac{m_{\oplus} \sqrt{GM_{\odot} a(1-e^2)}}{\hbar} = 2.523 \times 10^{74}, \quad (11)$$

the number of times the gravitational Bohr length $\hbar^2/(m_{\oplus}^2 GM_{\odot}) = 2.35 \times 10^{-138}$ m fits into an astronomical unit, square-rooted. [H12a, H12b] For an atom it is a few units. The unit is the same at every size (0.3a); the copies of the cone differ in how many units an orbit holds, and §7 shows which terms a small count brings forward.

6.3 The cutting plane, quantized

The two vectors that fix the cutting plane of Theorem 1, $\mathbf{L}$ for its height and $\mathbf{A}$ for its tilt, become operators at the atomic scale. Pauli symmetrized $\mathbf{A}$, scaled it to $\mathbf{M} = \mathbf{A}/\sqrt{-2mH}$ on bound states, and closed the algebra of $\mathbf{L}$ and $\mathbf{M}$ [11]: $\text{so}(4) \cong \text{su}(2) \oplus \text{su}(2)$, with equal Casimirs $j(j+1)\hbar^2$,

the symmetry Fock and Bargmann identified [12, 13]. The quantum form of (5), $\mathbf{A}^2 = m^2 K^2 + 2mH(L^2 + \hbar^2)$, then reads $-mK^2/(2H) - \hbar^2 = 4j(j+1)\hbar^2$, and

$$H = -\frac{mK^2}{2\hbar^2 n^2}, \quad n = 2j + 1 = 1, 2, 3, \dots, \quad \text{degeneracy } n^2 = \sum_{l=0}^{n-1} (2l + 1). \quad (12)$$

[H14a, H14b] With the nucleus held fixed $E_1 = -13.6057$ eV; with the reduced mass of hydrogen, -13.5983 eV. [H13b, H13f] A finite-difference solution of the radial equation for $l = 0, 1, 2$ returns $E_n = -1/(2n^2)$ hartree, the same for every l at one n , to 4×10^{-6} . [H14c] The conserved quantities that tilt the planes cutting the cone of the Solar System quantize the atom.

7 The cylinder: quantum mechanics in cylindrical form

7.1 The plane normal to the axis

A helix lives on a cylinder, and a point of it has cylindrical coordinates (z, ρ, φ) : the position z along the axis, the radius ρ , and the angle φ . Its projection onto the plane normal to the axis is the vector $\rho(\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2)$. Multiply on the left by $\mathbf{e}_1$ and it is the even element

$$\mathbf{e}_1 \rho(\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2) = \rho(\cos \varphi + \Sigma \sin \varphi) = \rho e^{\Sigma \varphi}, \quad \Sigma = \mathbf{e}_1 \mathbf{e}_2, \quad \Sigma^2 = -1. \quad (13)$$

Proposition 3 (The unit imaginary is the unit area of the transverse plane). *The map $u + iw \mapsto u + w\Sigma$ carries the complex numbers onto the even subalgebra of the plane's geometric algebra and complex multiplication onto the geometric product. [H15a, H15b] Multiplication by $e^{\Sigma\theta} = \cos \theta + \Sigma \sin \theta$ is the rotation of the plane by θ ; [H15c] complex conjugation is reversion, $\Sigma \mapsto -\Sigma$, the reversal of the sense of turning; [H15d] and the Hermitian product of two numbers is the geometric product of the two vectors,*

$$\overline{\psi_a} \psi_b \longleftrightarrow \mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \quad \mathbf{a} \wedge \mathbf{b} = (a_1 b_2 - a_2 b_1) \Sigma, \quad (14)$$

its real part the dot product and its imaginary part the oriented area. [H15e] Every complex equation of quantum mechanics is therefore a pair of real equations in the plane normal to the axis.

So $\psi = \rho e^{i\varphi}$ is the cross-section of a helix: ρ its radius, the A of I-4, and φ its phase angle. The axis is time for an observer inside the system and the direction of motion for an observer outside it (Theorem 3). The ψ -helix's advance per turn and its two rates are data; its apparent angle depends on the scale of ρ and is none.

7.2 The two observers of a pattern's phase

Theorem 3 (The de Broglie wave is the bookkeeper's reading). *In its own frame a closed pattern's phase only circulates, $\varphi = -\omega_0 \tau + \varphi_0$ with $\omega_0 = mc^2/\hbar$: a helix along its own time axis, and ψ is that helix projected onto the plane normal to the axis. For a bookkeeper in whose frame the pattern moves at v , the pattern's own time coordinate is $\tau = \gamma(t - vx/c^2)$, its clock's reading on its worldline, so at one bookkeeper time the phase advances along the path,*

$$\varphi = kx - \omega t + \varphi_0, \quad k = \frac{\gamma m v}{\hbar} = \frac{p}{\hbar}, \quad \omega = \frac{\gamma m c^2}{\hbar} = \frac{E}{\hbar}, \quad \lambda = \frac{2\pi}{k} = \frac{h}{p}, \quad (15)$$

with $\omega^2 = c^2 k^2 + \omega_0^2$, the strand's right triangle $E^2 = (pc)^2 + (mc^2)^2$ of 0.1d. The phase rate $\omega/k = c^2/v$ and the run $d\omega/dk = v$ multiply to c^2 (I-2).

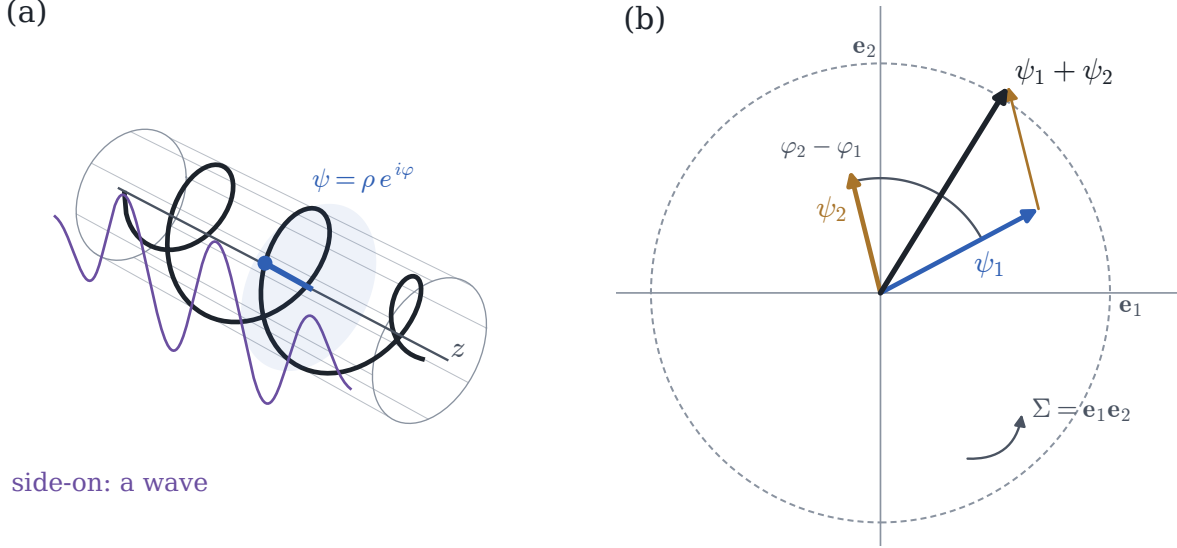


Figure 3: The cylinder and its cross-section. (a) A stretch of helix (z, ρ, φ) on its cylinder, with its cross-section at one z : the point $\psi = \rho e^{i\varphi}$ of the plane normal to the axis. Side-on, its shadow $\rho \sin \varphi$ is a wave. (b) The cross-section, spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, with unit bivector $\Sigma = \mathbf{e}_1 \mathbf{e}_2$, the unit imaginary. Two helices crossing one plane add as vectors, and the square of the sum is the law of cosines, $\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\varphi_1 - \varphi_2)$: interference.

Proof. Substitute τ into $-\omega_0\tau$ and read off the coefficients of x and t . [H16a, H16b] $d\omega/dk = c^2k/\omega = v$, so $(\omega/k)(d\omega/dk) = c^2$. [H16c, H16d] $\square$

De Broglie derived $\lambda = h/p$ this way in 1924, from the pattern's own clock and the relativity of simultaneity [23]. In the pattern's frame there is no wavelength: the phase is uniform in space and turns in time. A probe that measures the wavelength, a crystal, a grating or a pair of slits, is outside the system and moves relative to it, and it reads the helix. The record is the bookkeeper's: 54 eV electrons have $h/p = 1.669 \text{ \AA}$, and Davisson and Germer measured $1.65 \text{ \AA}$ [24]; [H16f] C_{60} molecules at 220 m s^{-1} have 2.5 pm and interfere accordingly [25]; [H16g] molecules of more than $25\,000 \text{ u}$ do the same [26]. An atom or a molecule moving through a laboratory is one helix in its centre-of-mass coordinate, of pitch h/P set by the momentum P of the whole: the wavelength these interferometers measure. The de Broglie helix is not a path. Theorem 2 applied to the rest circulation, of period $T_0 = h/(mc^2)$, advances $\gamma v T_0 = (\gamma v/c)^2 h/p$ per turn, $3.5 \times 10^{-4} \text{ \AA}$ for the 54 eV electron; [H16h] h/p is the spacing of equal phase along the path at one bookkeeper time.

A third observer turns with the pattern's own circulation. Writing $\psi = e^{-i\omega_0 t} \psi_S$ in the Klein-Gordon equation gives

$$i\hbar \partial_t \psi_S = -\frac{\hbar^2}{2m} \nabla^2 \psi_S + \frac{\hbar^2}{2mc^2} \partial_t^2 \psi_S, \quad (16)$$

and the last term drops for a slowly varying envelope: the Schrödinger equation is the helix read from a plane turning at the rest circulation ω_0 . [H17] That observer's phase rate is $\hbar k/2m = v/2$, while the run is v in every reading; [H16e] only differences of rates reach a probe.

7.3 The Schrödinger equation on the cylinder

Theorem 4 (Cylindrical form). *With $\psi = \rho e^{i\varphi}$, $\rho \geq 0$, the Schrödinger equation $i\hbar \partial_t \psi = -(\hbar^2/2m)\nabla^2 \psi + U\psi$ is, on every region where $\rho > 0$, the pair of real equations*

$$\partial_t(\rho^2) + \nabla \cdot \left(\rho^2 \frac{\hbar}{m} \nabla \varphi \right) = 0, \quad (C)$$

$$\hbar \partial_t \varphi + \frac{\hbar^2}{2m} |\nabla \varphi|^2 + U + Q_\rho = 0, \quad Q_\rho = -\frac{\hbar^2}{2m} \frac{\nabla^2 \rho}{\rho}, \quad (H)$$

continuity for the radius and Hamilton–Jacobi for the phase angle. The radius vanishes on the nodal set, and the phase angle closes:

$$\oint \nabla \varphi \cdot d\mathbf{l} = 2\pi n, \quad n \in \mathbb{Z}, \quad (W)$$

on every closed loop that avoids the nodal set, n being the number of turns the helix makes about the nodes the loop encircles.

Proof. Substituting $\psi = \rho e^{i\varphi}$ and multiplying by $e^{-i\varphi}$, the Schrödinger equation becomes $-\rho(H) + i(\hbar/2\rho)(C) = 0$, identically, where (H) and (C) denote the left sides; the real and imaginary parts vanish separately. [H18] ψ is single-valued exactly when φ returns to itself modulo 2π around every loop, which is (W). $\square$

(C) and (H) are Madelung’s equations [27]. Without (W) they admit solutions that no single-valued ψ has [28]. On the cylinder (W) is the statement that φ is an angle, so that a helix closes on its cylinder only after a whole number of turns. Each term has a geometric reading. ρ^2 is the share density, the squared projected radius of the Born-rule paper. $\hbar \nabla \varphi$ is the momentum: $|\nabla \varphi|$ is 2π over the helix’s local pitch. Q_ρ is the profile term: the confinement cost of I-5 written on the radius profile, larger for a tighter profile, as a shorter string has a higher mode. It enters the mean energy as $\langle Q_\rho \rangle = (\hbar^2/2m) \int |\nabla \rho|^2$, and the kinetic energy splits exactly into the share of the phase’s turning and the share of the profile.

7.4 Stationary states, and what shrinking brings forward

A stationary state has $\varphi = \varphi_s(\mathbf{x}) - Et/\hbar$: a helix along the time axis, turning at $E/\hbar$ in the frame of the Schrödinger observer. In the ground state of hydrogen the phase does not turn in space at all. With $\rho = e^{-r/a}$ and $\nabla \varphi = 0$, (H) holds at every r exactly when $a = \hbar^2/(mK)$ and $E = -mK^2/(2\hbar^2)$: the whole kinetic energy sits in the profile term. [H20a] In the $2p$ state with $m_l = +1$ the phase turns once about the z -axis, $\varphi_s = \phi_{az}$, the azimuth, (W) holds with $n = 1$, and the phase’s turning, the potential and the profile term add to $-1/8$ hartree at every point. [H20b, H20c]

Proposition 4 (The profile share). *In the circular state of principal number n ($l = |m_l| = n - 1$) of an inverse-square pair, the phase’s turning carries the fraction $(n - 1)/n$ of the kinetic energy and the profile term the fraction $1/n$; for $n = 1$ the turning share is zero and the profile carries all.*

Proof. With $\rho = r^l e^{-r/(na_0)} \sin^l \theta$, $l = n - 1$, and $|\nabla \varphi|^2 = l^2/(r \sin \theta)^2$, the turning share is $\frac{\hbar^2}{2m} l^2 \langle r^{-2} \sin^{-2} \theta \rangle$. With $\langle r^{-2} \rangle = 1/(n^3(l + \frac{1}{2})a_0^2)$ and $\langle \sin^{-2} \theta \rangle = (2l + 1)/(2l)$ for $|m_l| = l$, it is $\frac{\hbar^2}{2ma_0^2} l/n^3$ for $l \geq 1$, the fraction $l/n = (n - 1)/n$ of $\langle T_{\text{kin}} \rangle = -E_n = \hbar^2/(2ma_0^2 n^2)$; for $l = 0$ the factor l^2 makes it zero. The rest, $1/n$, is the profile share $\frac{\hbar^2}{2m} \int |\nabla \rho|^2$. Both are checked exactly for $n = 1, \dots, 6$. [H21a] $\square$

The profile share decides which reading shows. Read in its centre-of-mass coordinate as the circular state of the same equations, with $n \approx L/\hbar$, the Earth's orbit has a profile share of 4×10^{-75} , and (H) is the Hamilton–Jacobi equation of the pair, whose rays are the conic sections of Theorem 1: the flattened cone. A circular state at $n = 72$ has 1.4%; Maeda and Gallagher's packet at that n , built from np states, runs a highly eccentric, nearly linear Kepler orbit at the period $n^3 T_1$. The ground state of hydrogen has 100%: no orbit, no phase turning, all profile. [H21b] Shrinking the cone brings the profile term forward, and that is all it does.

8 Quantum phenomena on the cylinder

Each phenomenon below is a reading of Theorem 4 and Proposition 3, carried to two ports for spin, to a joint configuration for two probes, and to Dirac's equation for the strand at c .

Quantization is closure. On a ring of radius R , $\psi = e^{in\theta}$ is single-valued only for integer n ; it carries $L = n\hbar$ and $E = n^2\hbar^2/(2mR^2)$, and a half-integer winding fails to close, $e^{i\pi} = -1$. [H19a, H19b] (W) makes every winding about an axis an integer, $L_z = m_l\hbar$. The radial closure comes from the radius profile's boundary data through Q_ρ , and at large n the two closures combine into the Bohr–Sommerfeld rule with its half-turns.

The zero-point energy is the profile term. For the oscillator $U = m\omega^2 x^2/2$ the ground state $\rho = e^{-m\omega x^2/2\hbar}$ has $\nabla\varphi = 0$ and $Q_\rho = -m\omega^2 x^2/2 + \hbar\omega/2$, so $U + Q_\rho = \hbar\omega/2$ at every x . [H22] The zero-point energy is the confinement cost of the ground state's profile, and the omnium that carries it is never empty (0.1).

Standing waves are two helices of opposite hand. $\cos kx = (e^{ikx} + e^{-ikx})/2$: a standing wave is two helices turning in opposite senses on one cylinder. Their sum does not turn; it breathes along one diameter and vanishes where the two cancel. [H23] These are the nodal patterns of P-2: on their nodes the share density of the standing mode vanishes.

Interference is the law of cosines. Two helices crossing one plane add as vectors,

$$|\psi_1 + \psi_2|^2 = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\varphi_1 - \varphi_2), \quad (17)$$

and the cross term is the dot product of (14) (Figure 3b). [H24a] Behind two slits $\varphi_1 - \varphi_2 = 2\pi\Delta s/\lambda$, with $\lambda = h/p$ the bookkeeper's wavelength of Theorem 3. A mark that records the path multiplies the cross term by the overlap of the two marks; orthogonal marks set the two projections perpendicular, and the fringes go. [H24b] Single patterns build the fringes one at a time [29]: each seats whole in one port with the frequency of its share, by the seating law of the Born-rule paper [2].

Spectral lines are beats. Two stationary helices of rates ω_1 and ω_2 on one cylinder give a share density that beats at $\omega_2 - \omega_1$: the radiated frequency is the beat of two phase helices, [H25a] 10.1987 eV and 121.57 nm for the $2 \rightarrow 1$ line of hydrogen with the reduced mass, measured at 121.567 nm. [H25b] A jump between two levels, caught mid-flight, runs a continuous, coherent and deterministic course and can be reversed before it completes [30]. Every transition is an eigenmode relaxation of the atom's pattern within its braid class (I-5), a system-wide settlement of the state it starts from (0.1a).

The commutator is an oriented area. Shift a pattern along the axis by ξ , $S_\xi \psi(x) = \psi(x - \xi)$, and twist its helix by k , $W_k \psi(x) = e^{ikx} \psi(x)$. Then

$$S_\xi W_k = e^{-ik\xi} W_k S_\xi : \quad (18)$$

shifting and twisting differ by a turn of the cylinder through $k\xi$, the oriented area $\hbar k \cdot \xi$ of the loop in phase space divided by $\hbar$. [H26a] Its generator form is $[x, -i\hbar \partial_x] = i\hbar$: [H26b] the i says that the loop turns the plane normal to the axis, and $\hbar$ converts the loop's area into the angle. A stretch of helix of length Δx carries pitches spread over $\Delta k \geq 1/(2\Delta x)$, with equality for a Gaussian stretch, $\Delta x \Delta p = \hbar/2$: [H26c] the bandwidth of a finite helix.

Tunnelling: the wall becomes porous. The omnium is never still. Density is nowhere uniform: around every point a sphere holds both higher- and lower-density regions (0.1), every step is a system-wide settlement of the state it starts from (0.1a), and every mode carries its resident disturbance [2]. A barrier is therefore no sealed wall. The omnium's motion carries its low-density regions about, and where they line up across the barrier they open a density geodesic, a nodal channel of P-2, along which the pattern's energy flows. The wall becomes porous, and a pattern that meets it again and again eventually finds it open. On the cylinder the wall's porosity is read on the radius. Inside a barrier $U_0 > E$ the radius decays: where the turning term is negligible, (H) reads $E = U_0 + Q_\rho$, so $\rho''/\rho = 2m(U_0 - E)/\hbar^2$ and $\rho \propto e^{-\varkappa x}$, $\varkappa = \sqrt{2m(U_0 - E)}/\hbar$. [H27a] By the seating law, the frequency with which an alignment reaches depth x goes as the share density ρ^2 [2]: it falls as $e^{-2\varkappa x}$ in the body of the wall, faster for a heavier pattern and a higher wall, and at the exit it is the transmitted share. [H27e] By (C) the transmitted current J is the same at every depth, and the phase turns at $\varphi' = mJ/(\hbar\rho^2)$, the flow per share. In the body of an opaque barrier most of the share sits in alignments that reach part way and fall back, and the phase barely turns; at the exit every alignment that arrives spans the wall, and the phase turns at the outgoing rate k . The “imaginary momentum” $i\hbar\varkappa$ of the standard account is the radius gradient: $(-i\hbar \partial_x \psi)/\psi = \hbar\varphi' - i\hbar\rho'/\rho$, whose bivector part is $-\hbar\rho'/\rho$. The transmitted share through a barrier of width w is $[1 + U_0^2 \sinh^2(\varkappa w)/(4E(U_0 - E))]^{-1}$, as a transfer matrix returns; [H27b] on the exact state $\rho^2\varphi'$ is the same at every point of the barrier, $\varphi' = k$ at the exit, and the phase turns through 0.87 rad inside a barrier of $\varkappa w = 2.24$. [H27d] For a 4.5 eV barrier across the vacuum gap of a scanning tunnelling microscope, $2\varkappa = 2.17 \text{ \AA}^{-1}$: the current falls by a factor 8.8 per ångström [31]. [H27c]

Two ports: a three-sphere and the half-angle. A pattern meeting a probe with two ports has projections $(z_1, z_2) = (\rho_1 e^{i\varphi_1}, \rho_2 e^{i\varphi_2})$ with $\rho_1^2 + \rho_2^2 = 1$: a point of the unit three-sphere. Its common phase runs along a Hopf fibre, which the probe does not resolve (I-4), and the Hopf map carries it to a point of the two-sphere, $\mathbf{n} = (2 \text{Re } \bar{z}_1 z_2, 2 \text{Im } \bar{z}_1 z_2, |z_1|^2 - |z_2|^2)$, unchanged by the common phase. [H28a, H28b] The shares $|z_1|^2$ and $|z_2|^2$ are port shares, never an α (I-6). A closed pattern of the nontrivial 2π -rotation class, turned by θ about the probe's axis, turns each port's projection by $\theta/2$, so a turn by 2π sends (z_1, z_2) to $(-z_1, -z_2)$, half way round its fibre: [H28c] the 4π periodicity measured with neutron interferometers in 1975 [32, 33]. A Stern–Gerlach field at angle χ reads the shares $\cos^2(\chi/2)$ and $\sin^2(\chi/2)$. [H28d] That a closed pattern's 2π -rotation class is the nontrivial one is to be derived (D-03; core O-11); once it is, the identity $(8\pi G/c^4) F_{\max} = 2\pi$ fixes its character to -1 with no fitted parameter (I-5).

One pattern at two probes. A state read at two probes that is not a product has a radius and a phase angle on the joint configuration that do not split into one for each probe: the pair carries

one helix, the shared bivector grade of I-4. The singlet is unchanged by every common rotation, $(R \otimes R)\Psi = \Psi$, [H29d] and its correlation along two analyser axes is $C(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} = -\cos \theta_{ab}$, the projection geometry of one shared rotor; [H29a] with the four analyser axes spaced by $\pi/4$ it reaches $|S| = 2\sqrt{2}$ in the CHSH sum. [H29b] Each local state is $I/2$ whatever the distant analyser does: no signal passes. [H29c] Loophole-free tests violate Bell's inequality as quantum mechanics predicts for the states they prepare: $S = 2.42 \pm 0.20$ for electron spins [34], and the Eberhard inequality for photon pairs [35, 36]. Bell's inequality premises that the outcomes factorize into two local responses, one per probe. FDT reads every event as one system-wide settlement of the state it starts from (0.1a), and the pair is one pattern with a shared bivector grade (I-4). The factorization premise is the particle-by-particle reading the core declines. The one settlement fixes both seatings from the state it starts from, both analysers included, while each local share stays blind to the distant analyser, and the measured violation is accepted in full.

Two senses of turning, and the strand at c . The two roots $\pm\omega$ of $\omega^2 = c^2 k^2 + \omega_0^2$ are the two senses in which a helix turns. P-4 reads matter and antimatter as opposite handedness of one configuration. In Dirac's equation the velocity operator $c\gamma^0\gamma^k$ has the eigenvalues $+c$ and $-c$ only [37, 38]: [H30a] the instantaneous run of the electron's strand is c , and its mean is v , as 0.1d reads every closed pattern's strand, $v^2 + u^2 = c^2$ [3]. Written in the real algebra of spacetime, the unit imaginary of Dirac's equation is the bivector $\gamma_2\gamma_1$ of the spin plane [39, 40], which squares to -1 and commutes with γ_0 and γ_3 : [H30b] the plane normal to the axis once more.

Measurement is phase fixation. A probe seats the pattern whole in one port and keeps its projection onto the seated port's mode; the frequency of each port is its share (I-4; the seating law of [2]). What the probe does not resolve is the helical phase φ_0 of the resident disturbance; the rate kernel by which the probe's coupling advances the delivered fraction is to be derived (D-04; core O-12).

9 Declined readings and convergence

The standard account reads ψ as a probability amplitude and i as an imaginary unit with no counterpart in nature. FDT reads ψ as the projection of a helix onto the plane normal to its axis and i as the oriented unit area of that plane (I-4, Proposition 3).

The standard account gives matter a dual nature, particle and wave. FDT reads one pattern, whose de Broglie wave is the bookkeeper's reading of its own circulation (0.1c, 0.1d, Theorem 3).

The standard account imposes single-valuedness and boundary conditions on a complex amplitude, and quantization follows from the conditions. FDT reads closure: φ is the angle of a helix, which closes on its cylinder only after a whole number of turns, at every size, and at the atomic size the action per turn is a few $\hbar$ (0.1a, 0.3a, Theorem 4).

The standard account sends a tunnelling particle across a forbidden region with negative kinetic energy and imaginary momentum. FDT reads a wall made porous by the omnium's motion, whose low-density regions line up into a density geodesic for the pattern's energy; on the cylinder the radius decays, and the phase turns only as fast as the transmitted current requires (0.1, 0.1a, I-4).

The standard account reads the zero-point energy of an unexcited mode as the fluctuation of an empty vacuum. FDT reads the profile term of the mode's ground state, the confinement cost of I-5, carried by the resident disturbance of an omnium that is never empty (0.1; [2]).

Dirac's hole theory fills a sea with negative-energy states. FDT reads the two signs of ω as the two hands of the helix, and matter and antimatter as opposite handedness of one configuration

(P-4).

The standard account has entangled particles act on each other at a distance, or yield outcomes that are irreducibly random. FDT reads one pattern with a shared bivector grade, each event a system-wide settlement, with the local shares blind to the distant analyser (0.1a, I-4).

The standard account collapses the wavefunction at measurement. FDT reads phase fixation: the probe seats the pattern whole and keeps its projection onto the seated mode (I-4).

The standard account treats planetary orbits by classical mechanics and atomic orbits by quantum mechanics, joined by a correspondence limit. FDT reads one cone at every size, whose profile share, $1/n$, decides which terms show (0.3a, Proposition 4).

Convergence. For the same sources and data FDT returns: Kepler’s three laws for the relative orbit, exactly in model on the pair domain (I-1, I-4 L1); the perihelion advance and the deflection of light of general relativity at leading order (I-3, L2), $42.98''$ per century for Mercury and $1.75''$ at the solar limb; Rutherford’s cross section, which the classical hyperbolas and the quantum Coulomb pair share; the gross structure of the hydrogen spectrum with the measured constants and the reduced mass; every solution of the Schrödinger equation, of which Theorem 4 is an exact rewriting on $\rho > 0$, with (W) and $\rho e^{i\varphi}$ smooth across the nodal set; the de Broglie wavelengths of electrons, atoms and molecules; interference fringes; tunnelling rates; the statistics of two-port patterns, and their 4π periodicity with the class of D-03; and the correlations $-\cos\theta$ of the singlet, with Tsirelson’s $2\sqrt{2}$ and no signalling. No divergence is claimed: the paper supplies what ψ is, not a changed observable.

10 What ψ is

The density coordinate draws a cone, and the cone is the light cone. Every orbit about a source is the plane in which the source’s light cone meets the orbit’s world-tube; pressed flat, the plane is Kepler’s conic, and the cone’s own coordinate gives Kepler’s third law. The worldline winds on the world-tube as a helix. An observer inside the system projects it along the system’s time and sees the closed orbit; an observer outside projects it along a time in which the system moves and sees the helix, the giant body moving in a wave. The atom is the same cone at the size where an orbit holds a few units of $\hbar$ per turn. There the helix’s closure shows, and so does the profile term of its radius. ψ is the cross-section of that helix, the point where it crosses the plane normal to its axis. For an observer inside the system the axis is time, and ψ is a point turning in time; for a bookkeeper outside it the axis is the direction of motion, and ψ is the de Broglie wave. Its radius is the square root of a share density, its angle is the phase, and its i is the oriented area of the plane it lies in. Restore the axis and quantum mechanics is the geometry of a helix on a cylinder.

11 Status register

result	rests on	status
The cone of α ’s circles, truncated at the fold (§3)	0.3, 0.3a, P-4, I-6	derived
The plane section, e as slope, ℓ as height (Thm. 1)	I-1, I-4 L1	derived
The parabola at the flow speed $c\sqrt{\alpha_{\text{ext}}}$ (Cor. 1)	0.1c, 0.2	derived
The other nappe; Rutherford’s cross section (Cor. 2)	I-1 (inverse-square form); K from the record (D-01); sign by P-4	derived

result	rests on	status
Kepler I–III in the cone coordinate, relative orbit (Prop. 1)	I-1, I-4 L1	derived, exact in model
Precession $3\pi\alpha_{\text{ext}}(\ell)$ per revolution (Prop. 2)	I-3, I-4 L2	derived by containment
A stroke bends twice the flattened cone’s hyperbola taken to c	I-3, 0.1c	derived
One worldline, two readings (Thm. 2)	0.1c, 0.1d	derived
The Earth’s helix about the Sun’s track	Thm. 2; record	derived
The de Broglie helix as no path	Thms. 2, 3	derived
Shrinking changes only the action per turn, $L/\hbar$	0.1a, 0.3a	derived
The hydrogen spectrum from L and A (Pauli)	I-4; record	derived
i as the unit bivector of the transverse plane (Prop. 3)	I-4	derived
The de Broglie wave as the bookkeeper’s reading (Thm. 3)	0.1c, 0.1d, I-2, I-4	derived
Schrödinger as the co-rotating envelope	0.1d, I-4	derived
Cylindrical Schrödinger (C), (H), (W) (Thm. 4)	I-4, I-5	derived
The profile share $1/n$ (Prop. 4)	Thm. 4	derived
Closure, zero point, standing waves, interference, beats, commutator, bandwidth, tunnelling (§8)	Thm. 4, Prop. 3	derived
The porous wall: low-density regions aligned into a density geodesic; their reach goes as ρ^2	0.1, 0.1a, I-4; the seating law [2]	derived
Two ports on the three-sphere	I-4, I-6	derived
The singlet as one pattern at two probes; no signalling	I-4, 0.1a	derived
Dirac’s velocity eigenvalues $\pm c$, read as the strand at c	0.1d, I-5	derived
α labels for an atomic pair: the map from a charge to an I-1 input	I-1	to be derived (D-01; core O-6)
Masses, the confinement operator, the dimensionless content of K	I-5	to be derived (D-02; core O-9)
The nontrivial 2π -rotation class of a closed pattern; the half-angle and the 4π periodicity	I-5	to be derived (D-03; core O-11)
The seating rate kernel of a probe	I-4	to be derived (D-04; core O-12)
The aperture of the density cone near the fold	P-4	to be derived (D-05; core O-8)

invariant	where it enters
I-1	the pair law on the cone, $F = F_{\text{max}}\alpha_1\alpha_2 = GMm/r^2$, exact in model; the inverse-square form behind every plane section (Thm. 1); one equation with four names for the atomic pair
I-2	one carrier: the cone’s generators are the stroke lines at $\pi/4$; a closed pattern’s phase rate and run multiply to c^2 (Thm. 3)
I-3	the field equations, unchanged: the precession $3\pi\alpha_{\text{ext}}(\ell)$ and the factor of two in the bending of a stroke; containment
I-4	the quantum face: ψ the projection of a helix onto the plane normal to the cone axis; phase ignorance; determinism; the shared bivector grade of one pattern read at two probes
I-5	matter as closed stroke: the electron and the proton as one $n = 3$ braid in two rooms; the electron’s strand at c ; the profile term as the confinement cost; transitions as eigenmode relaxations; masses and mass ratios stay with the operator (D-02; core O-9)
I-6	the open interval: the fold is a circle, $\alpha_{\text{ext}} < 1$ strictly; port shares are not α
I-7	every energy of the Sun enters $\mathcal{M}_{\odot}$ once, through the measured $G\mathcal{M}_{\odot}$, and fixes R_s

A Receipt

The script `receipt_psi.py`, in the folder of this paper, checks the displayed identities symbolically with SymPy and the displayed numbers numerically, with independent numerical checks where a closed form could hide an error; its output is `receipt_output.txt`. Run `python3 receipt_psi.py`. Result: **96/96 PASS**.

check	content
H01a-c	$\alpha_{\text{ext}}(z) = R_s/z = \kappa(z)/\kappa(R_s)$; scale invariance; $F_{\text{max}}\alpha_1\alpha_2 = G\mathcal{M}m/r^2$
H02a-e	conic lifted to $z = r$ lies on $z = \ell - ex$; repulsive branch on $z = -r$; $\mathbf{A} \cdot \mathbf{r} = L^2 - mKr$; $ \mathbf{A} ^2 = m^2K^2 + 2mEL^2$; $\mathbf{A} \perp \mathbf{L}$
H03a-c	integrated ellipse, parabola and hyperbola: the lift lies on one plane to 10^{-11}
H04	integrated orbit: areal rate constant to 2×10^{-12}
H05a-d	$e = 1 \Leftrightarrow E = 0$; perihelion speed over flow speed $\sqrt{(1+e)/2}$; vis-viva at perihelion; parabola at $(v/c)^2 = \alpha_{\text{ext}}$
H06a-c	$\sin(\delta/2) = 1/e$; $b = (K/2E) \cot(\Theta/2)$; Rutherford's cross section
H07a-d	vis-viva in α_{ext} ; $T^2 = 8\pi^2 a^3 / (c^2 R_s)$; $E/(mc^2) = -\alpha_{\text{ext}}(a)/4$; the period on an ellipse
H08a-e	$R_s = 2953.25$ m; α_{ext} at 1 AU and at the solar surface; 29.78 km s^{-1} ; 365.26 d; the relative orbit, 1.5×10^{-6}
H09a-b	Mercury 5.019×10^{-7} rad and $42.98''$ per century; integrated relativistic orbit
H10	$1.75''$ for the stroke, $0.88''$ for a hyperbola at c
H11a-g	the Sun's 248.5 km s^{-1} ; the ecliptic pole and the 30.4° tilt; 52.4 AU per turn; 5.6° to 6.9° ; the worldline at $v/c = 9.94 \times 10^{-5}$; the Lorentz advance γVT and period γT ; the full solar motion: 248.9 km s^{-1} , 29.4° , 52.5 AU
H12a-b	$L_\oplus/\hbar = 2.523 \times 10^{74}$; gravitational Bohr length
H13a-f	a_0 , E_1 , v_1 , T_1 ; 17.5 GHz is $n = 72$; 274.3 nm , 30.38 km s^{-1} , 56.73 ps ; E_1 with the reduced mass, -13.5983 eV
H14a-c	SO(4) Casimir gives (12); degeneracy n^2 ; finite-difference spectrum
H15a-e	$\Sigma^2 = -1$; complex product as geometric product; rotor; conjugation as reversion; Hermitian product as $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$
H16a-h	de Broglie from $\tau = \gamma(t - vx/c^2)$; $\omega^2 = c^2 k^2 + \omega_0^2$; phase rate $\times$ run $= c^2$; run $= v$; co-rotating phase rate $v/2$; $1.669 \text{ \AA}$; 2.5 pm ; the rest circulation's path advances $(\gamma v/c)^2 h/p$ per turn
H17	Klein-Gordon with $\psi = e^{-i\omega_0 t} \psi_S$ gives (16)
H18	Schrödinger $= -\rho(H) + i(\hbar/2\rho)(C)$, identically, in three dimensions
H19a-b	ring: single-valued for integer n , $L = n\hbar$, $E = n^2 \hbar^2 / (2mR^2)$; half-integer fails
H20a-c	hydrogen ground state in (H); $2p$, $m_l = +1$: $-1/8$ hartree at every point; one turn about the axis
H21a-b	circular states $n = 1, \dots, 6$: profile share $1/n$; 4×10^{-75} , 1.4%
H22	oscillator: $U + Q_\rho = \hbar\omega/2$
H23	standing wave as two counter-turning helices
H24a-b	law of cosines; a path mark multiplies the cross term by the marks' overlap
H25a-b	beats at $\omega_2 - \omega_1$; 10.1987 eV , 121.57 nm
H26a-c	$S_\xi W_k = e^{-ik\xi} W_k S_\xi$; $[x, -i\hbar\partial_x] = i\hbar$; Gaussian $\Delta x \Delta p = \hbar/2$
H27a-e	barrier: $\varkappa$ from (H) where the turning term is negligible; exact transmission against a transfer matrix; $2\varkappa = 2.17 \text{ \AA}^{-1}$, $\times 8.8$ per $\text{\AA}$; on the exact state $\rho^2 \varphi'$ constant, $\varphi' = k$ at the exit, 0.87 rad ; in an opaque barrier ρ^2 falls as $e^{-2\varkappa x}$ and is T at the exit
H28a-d	Hopf map onto the unit two-sphere; common phase along the fibre; 2π turn gives -1 for the class of D-03; $\cos^2(\chi/2)$
H29a-d	singlet $C = -\cos\theta$; CHSH $2\sqrt{2}$; local state $I/2$; $(R \otimes R)\Psi = \Psi$
H30a-b	Dirac velocity eigenvalues $\pm c$; $(\gamma_2 \gamma_1)^2 = -1$, commuting with γ_0 and γ_3
H31	$F_{\text{max}} d_{\text{min}} t_{\text{min}} = \hbar$

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