

Collective geometry and the Yang–Mills gap

The physical domain of the mass-gap problem in Fundamental Density Theory

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Result. The Clay problem asks for a quantum Yang–Mills theory on the fixed Minkowski background $\mathbb{R}^4$, with a unique Poincaré-invariant vacuum of zero energy and a mass gap. Fundamental Density Theory retains the Einstein field equation unchanged, $G_{\mu\nu} = \kappa T_{\mu\nu}$ (I-3), and has no background metric: the adjustment of clocks and rulers is the geometry (0.1a). On a fixed flat background $G_{\mu\nu} \equiv 0$, so I-3 admits only states whose total source vanishes; the one such state is the empty state, and the omnium has no empty state (0.1). **If general relativity is ignored, the Clay problem statement does not apply to reality** (Theorem 5). Along any family of complete patterns with fixed physical metric the total source obeys $d(\kappa T_{\mu\nu})/ds = 0$ (Theorem 7); the pure-gauge interpolation $A = s\theta$ on a fixed metric carries a nonzero source at every s between its two pure-gauge ends, which a flat background does not hold, and at $s = 1/2$ it fails this condition at second order as well, so the negative second derivative of its magnetic energy there is a property of an inadmissible family, not a property of the omnium (Proposition 9). Every admissible region carries positive energy (Proposition 12); the lowest pattern of a closed pattern’s own region cycles with period h/E_{base} (Proposition 15); on a spatial geometry of infinite volume, the volume the Clay $\mathbb{R}^3$ posits, the energy measure of the omnium is infinite with the floor of the spacetime phase (Theorem 13, D-3), so no subtraction of a constant relates it to a zero-energy vacuum. The gap statement on the physical domain is the uniform estimate (9), whose lower edge is the step between the two lowest confinement eigenvalues of a closed pattern’s own region, where the one carrier stands (I-2, I-5); its derivation is D-1. FDT recovers the fixed-background equations and their classical theorems as the $\alpha_{\text{ext}} \rightarrow 0^+$ reading of the coupled solution and declines the endpoint $R_s = 0$, at which the Clay statement is posed and no branch of the atlas exists (I-6).

0 Conventions and provenance

This document is self-contained on the locked FDT core v6.3 (CORE-FDT-INVARIANTS.md, author lock 2026-10-26), which it quotes by section: 0.1, 0.1a, 0.1b, 0.2, 0.3, 0.3a, 0.4, 0.5, 0.6, 0.7, 0.8, I-1 to I-7 with I-3a, I-5a and I-5b, P-2, and the core’s register items O-4, O-6, O-9, O-10, O-13, O-14 and O-15. Every statement below is of one of three kinds: a quotation of the core; a derivation from the core and displayed mathematics; or an item marked to be derived (D-1, D-2, D-3) and collected in §9. The consensus record enters through the Clay problem statement [1] and the axioms it names [2, 3], the classical fixed-background well-posedness theorems [4, 5], the measured hadron spectrum [8], the glueball spectra computed in lattice gauge theory [6, 7], and the measured temperature and anisotropy of the standing-wave state [9, 10, 11].

Notation. $\kappa = 8\pi G/c^4$ (0.2). The total source is the one stress tensor $T_{\mu\nu}^{(\Omega)}$ of the omnium, every channel entering once (0.5, I-7); it is written $T_{\mu\nu}$. $\mathcal{M}$ is the total gravitating energy (0.5) and $R_s = 2G\mathcal{M}/c^2$. E_{base} is the lowest eigenvalue of the generator of a closed pattern’s own region (Proposition 11) and nothing else. The number of dimensions is $3 + 1$ throughout. A complete physical pattern is the omnium in one configuration,

$$z = (g, T_{\mu\nu}, \text{atlas data, phases, closure labels}, \dots) \in \mathcal{D}_F,$$

described by its metric g (0.1a: the pattern’s adjustment is the geometry), its total source (0.5), its density on the branch-tagged atlas (0.3), its helical phases (I-4) and its integral closure labels (I-5, I-5a). The admissible domain $\mathcal{D}_F$ retains the chain $E/m = c^2 = 1/(\epsilon_0\mu_0) = (d/t)^2$ (0.1), the floor $d_{\text{min}} = 2\ell_P$, $t_{\text{min}} = 2t_P$ (0.1a), the open interval $0 < \alpha_\sigma < 1$ (I-6) and the simultaneity of the chain’s terms (0.1a). An *admissible region* is a

region of the physical spatial geometry that contains a ball of diameter $d_{\min}$ in the physical metric; nothing below the floor is a region of anything (0.1a, 0.1b).

1 Where the seven invariants enter

One substance (0.1, 0.1a, 0.1b). There is one substance, the omnium; matter, energy and spacetime are its readings, tied by the chain $E/m = c^2 = 1/(\epsilon_0\mu_0) = (d/t)^2$, and no variable of the chain vanishes. The four terms adjust simultaneously: a change of energy is a change of the distances and durations the pattern keeps, and the metric's response to $T_{\mu\nu}$ is that simultaneity at the scale of bodies (0.1a). There are no boxes: a confining domain is the pattern's own support with the pattern's own boundary data (0.1b). Every physical length and time belongs to a pattern, to a pattern-to-probe measurement, or to the floor fixed by $F_{\max} d_{\min} t_{\min} = \hbar$ (0.1a).

The force (I-1). $F = (c^4/4G) \alpha_1 \alpha_2$ at every scale. What the consensus calls the strong interaction is this law evaluated at colour-seat inputs; the maps from a colour seat to an I-1 input are the core's register item O-6. Density is nowhere uniform, so interaction pairs are available at every point and I-1 has no force-free point (0.1); (15) is that statement on admissible regions, and it is why the lowest pattern is never force-free and carries the network of §7. The Yang–Mills stress is a channel of the one total source $T_{\mu\nu}^{(\Omega)}$ (0.5, I-7), not a field beside the medium.

The carrier (I-2). photon = graviton = gluon = phonon = magnon: one spin-1 excitation of the density field, read in different density and geometry regimes. The identity is the chain read as carriers: the chain gives one speed to every reading, and 0.1b leaves one medium for them to be disturbances of (0.1, 0.1b). The gluon is the stroke standing inside a closed pattern's own region. The standard account reads the Yang–Mills field as independent gauge quanta propagating on an empty background. FDT reads it as the one carrier held by topological closure (I-5), and the mass gap as the step between the two lowest costs of the closed region in which that stroke stands.

The field equation and the atlas (I-3, 0.3, 0.3a). $G_{\mu\nu} = \kappa T_{\mu\nu}$ is unchanged, with $\kappa = 8\pi G/c^4$ displayed and $T_{\mu\nu}$ the total source; the density coordinate is downstream of the solution and branch-tagged. *Chart inventory.* Every α in this document is the A-1 compactness coordinate (0.4); no β_σ (A-2), no A-3, A-4 or A-5 token and no saturation diagnostic is used. The metric response of a configuration is read on the exterior branch of its source, $\alpha_{\text{ext}}(r_f) = R_s/r_f$ with $R_s < r_f < \infty$; the confined reading is carried by the configuration's interior chart, its skin at the fold and its skin radius taken from data; no transfer map between the two charts is displayed and none is used. On the A-1 chart the seam $\alpha_\sigma = 1/2$, at $r_f = 2R_s$ on the exterior branch and $r_f = R_s/2$ on the interior branch, is the Clifford torus of the atlas, splitting its support into the general-relativistic solid torus $\alpha_\sigma < 1/2$ (the spacetime phase, decompressed omnium) and the quantum-mechanical solid torus $\alpha_\sigma > 1/2$ (the matter phase, compressed omnium, closed braids) (0.3a). The three-sphere is the shape of the atlas, never a container the omnium occupies, and the two solid tori are two phases of one substance, never two substances (0.3a, 0.1). The family parameter s of §4 is a label on an auxiliary family; it is not an α value and carries no chart.

The quantum face (I-4). Quantum mechanics is helical geodesics on truncated density cones. The standard account reads ψ as a probability amplitude over the spectrum of a generator. FDT reads $\psi = Ae^{i\varphi}$ as the projection of a helix, its phase the helical phase and the dynamics deterministic (I-4), and accepts the spectral data of a generator—the mode costs $E_q = \hbar\omega_q$ (0.1a)—in full. The quadratic-form estimates below are statements about mode costs.

Mass as a confinement eigenvalue (I-5, I-5a, I-5b). For a pattern confined to its own region Ω_p with boundary data B_p , $D(\Omega_p, B_p; n, \chi, t) u_q = \omega_q u_q$ and $m_q c^2 = \hbar\omega_q$. The Yang–Mills gap is the step from the lowest mode cost of a closed pattern's own region, $E_{\text{base}} = \hbar\omega_0$, to the next: $\varepsilon_* = \hbar(\omega_1 - \omega_0)$, a difference of

confinement eigenvalues of the region in which the carrier stands. The support radius of a closed pattern is fixed by geometry, never by the mass it predicts (the vacuity guard). Closure labels are integral and discrete (I-5a) and a variation never differentiates them. The admissible sector of the gap problem is the closure sector of I-5a, $\ell_{\text{total}} \in K \cap \mathbb{Z}^2$ (closure to white), whose seat-lattice premises are the core's register item O-10. A configuration that exists on its own is a closure (I-5a), so every transition runs from closures to closures and keeps every exact book of I-5b: the sector is never left.

The open interval (I-6). $0 < \alpha_\sigma < 1$ on every branch; $\alpha_\sigma \rightarrow 0^+$ and $\alpha_\sigma \rightarrow 1^-$ are one-sided limits, never attained, and no closed value $\alpha_\sigma = 0$ or $\alpha_\sigma = 1$ exists. Minkowski space is $R_s = 0$: no source, no fold and no branch of the atlas; the exterior branch approaches $\alpha_{\text{ext}} \rightarrow 0^+$ and never arrives.

Density equivalence (I-7, 0.5). $\mathcal{M} = M + E/c^2 + PV/c^2 + aT^4V/c^2 + \dots$; every channel enters once; gravitational binding energy is read from the solved geometry and never added as a local term. The energy of a pattern's lowest state is a channel of $\mathcal{M}$ and sources curvature.

Nodal architecture (P-2, 0.8). The omnium is a vibrating elastic medium; the loci where the transverse density gradient vanishes are its density geodesics, into which excitations lock. The substrate's cumulative standing-wave state fills a universal domain, and each domain is a bounded configuration read on the same atlas as a nucleus (0.8). Every FDT spectral statement is posed on a bounded support: the gap on a closed pattern's own region (I-5), which lies within a universal domain (0.8); the Clay $\mathbb{R}^3$ is neither.

2 The fixed background admits only the empty state

Definition 1 (the Clay data). The Clay problem [1, pp. 5–6] asks: for every compact simple gauge group G , prove that a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$, existence including axiomatic properties at least as strong as those of [2, 3]. Its data are a Hilbert space $\mathcal{H}$ carrying a unitary representation of the Poincaré group of Minkowski space $(\mathbb{R}^4, \eta)$, with H and $\mathbf{P}$ the generators of time and space translations; positive energy, $0 \leq H$ [1, p. 5], which the axioms of [2] state as the spectrum condition, the joint spectrum of $(H, \mathbf{P})$ in the closed forward cone; a vacuum vector Ω_η , invariant under the Poincaré group and unique up to a phase [1, p. 5], the only vector invariant under translations [2], with $H\Omega_\eta = 0$ [1, p. 6]; and the gap: H has no spectrum in $(0, \Delta)$ for some $\Delta > 0$, and the supremum of such Δ , the mass of the theory, is finite [1, p. 6]. The metric η is fixed: it enters through Poincaré covariance and is not a variable of the theory. The problem is posed with the response of geometry to energy switched off.

Theorem 2 (the background forces a vanishing source). *Let I-3 hold: $G_{\mu\nu}[g] = \kappa T_{\mu\nu}$ with $\kappa = 8\pi G/c^4 > 0$ and $T_{\mu\nu}$ the total source. Every configuration whose geometry is η has $T_{\mu\nu} \equiv 0$.*

Proof. $G_{\mu\nu}[\eta] \equiv 0$, and $\kappa \neq 0$. □

Corollary 3 (every state but the vacuum fails I-3 on the background). *No state $\psi \in \mathcal{H}$ with $\psi \notin \mathbb{C}\Omega_\eta$ is a configuration of geometry η under I-3.*

Proof. Let P_η project onto $\mathbb{C}\Omega_\eta$ and let $\|\psi\| = 1$. By the spectrum condition a vector with $H\psi = 0$ has $\mathbf{P}\psi = 0$ and is invariant under translations, so it lies in $\mathbb{C}\Omega_\eta$ [2]: $\ker H = \mathbb{C}\Omega_\eta$. For $\psi \notin \mathbb{C}\Omega_\eta$ in the form domain of H , $(I - P_\eta)\psi \neq 0$ and $E = \langle \psi, H\psi \rangle = \|H^{1/2}(I - P_\eta)\psi\|^2 > 0$; outside the form domain the energy is infinite and the conclusion holds a fortiori. The gap sharpens the bound to $E \geq \Delta\|(I - P_\eta)\psi\|^2$; the conclusion holds for every theory on the background, gapped or not. This energy is a channel of the total gravitating energy (0.5): $\mathcal{M} \geq E/c^2 > 0$, $R_s = 2G\mathcal{M}/c^2 > 0$, and on the exterior branch $\alpha_{\text{ext}}(r_f) = R_s/r_f \in (0, 1)$ at every field radius $R_s < r_f < \infty$ (0.3). The geometry of the state is not η ; assigning η to it, as the Clay data do, contradicts Theorem 2. □

Proposition 4 (the omnium has no empty state). *No configuration of the omnium has $T_{\mu\nu} \equiv 0$.*

Proof. A configuration with $T_{\mu\nu} \equiv 0$ has $E = 0$ and $m = 0$: two variables of the chain vanish, which 0.1 excludes. The same configuration has $\mathcal{M} = 0$ and $R_s = 0$, so it has no fold and lies on no branch of the atlas: the exterior branch $R_s < r_f < \infty$ carries $\alpha_{\text{ext}} \rightarrow 0^+$ only as a one-sided limit, never attained (I-6). $\square$

Theorem 5 (the Clay statement and reality). *No state of the Clay theory, as posed on the fixed background, is a physical configuration. The states outside $\mathbb{C}\Omega_\eta$ fail I-3 on the background; the vacuum Ω_η , with $H\Omega_\eta = 0$, has zero total source, $\langle \Omega_\eta, T_{\mu\nu}(x)\Omega_\eta \rangle = 0$ at every point, and is the empty state, which the omnium does not have; its stress does not vanish as an operator, $T_{\mu\nu}(f)\Omega_\eta \neq 0$ for every nonzero stress $T_{\mu\nu}(f)$ smeared over a bounded region [2], and read that way the vacuum fails I-3 on the background with every other state. The Clay statement holds the metric fixed while the energy of its states varies: it ignores general relativity by construction. If general relativity is ignored, the Clay problem statement does not apply to reality.*

Proof. Corollary 3 and Proposition 4 exhaust $\mathcal{H}$ with the stress read as an expectation; read as an operator, the stress of every state is nonzero and Theorem 2 excludes them all. $\square$

The standard account reads Ω_η as the vacuum: empty space, the zero of energy, invariant under every translation. FDT reads the lowest state as the lowest pattern of the omnium on its own support (0.1b), with $E_{\text{base}} > 0$ that enters $\mathcal{M}$ once and sources curvature (0.1, 0.5, I-3); the subtraction that sets its energy to zero discards a gravitating channel and is available only where the response of geometry has already been switched off.

The standard account reads η as the arena on which the field's clocks and rulers are read. FDT reads clocks and rulers as closed patterns adjusting their own dimensions to keep the chain; their adjustment is the geometry, and there is no background metric beneath the patterns (0.1a).

Remark 6 (the classical record). The fixed background is not excluded by a classical singularity. On Minkowski space the Yang–Mills evolution is globally well-posed for finite-energy data: Klainerman–Machedon [4], and Oh [5, §1.8, Main Theorem] for constraint-satisfying data with $\bar{A}_i \in \dot{H}^1 \cap L^3$ and $\bar{E}_i \in L^2$, the admissible solution extending globally and uniquely in the temporal gauge. FDT has no blow-up at any scale: force and pressure are capped by F_{max} and no variable of the chain vanishes (0.8). The background is excluded by I-3 (Theorem 2), not by a divergence. Classical continuation of an evolution and the lower edge of a spectrum are different questions; neither implies the other.

3 Compatibility along fixed-metric families

Theorem 7 (compatibility conditions). *Let $s \mapsto z(s) \in \mathcal{D}_F$ be a differentiable family of complete solutions of I-3, represented on a common manifold in a consistent coordinate gauge, with metric $g(s)$ and total source $T_{\mu\nu}(s)$. Then, pointwise,*

$$DG_g[\dot{g}]_{\mu\nu} = \kappa \dot{T}_{\mu\nu} + \dot{\kappa} T_{\mu\nu}. \quad (1)$$

If the physical metric is fixed along the family, $\dot{g} = 0$, then

$$\frac{d}{ds}(\kappa T_{\mu\nu}) = 0, \quad (2)$$

and if it is fixed through second order,

$$\kappa \ddot{T}_{\mu\nu} + 2\dot{\kappa} \dot{T}_{\mu\nu} + \ddot{\kappa} T_{\mu\nu} = 0. \quad (3)$$

With the core constants fixed, $\dot{\kappa} = \ddot{\kappa} = 0$, and (2), (3) read $\dot{T}_{\mu\nu} = 0$, $\ddot{T}_{\mu\nu} = 0$ pointwise.

Proof. Differentiate $G_{\mu\nu}[g(s)] = \kappa(s)T_{\mu\nu}(s)$ once and twice in s ; set $\dot{g} = 0$, then $\ddot{g} = 0$. The product rule displays the general identity and introduces no variation of the constants; with $\kappa = 8\pi G/c^4$ fixed the two conditions are the vanishing of the first and second s -derivatives of the total source at every point. $\square$

Three readings fix the meaning of Theorem 7. First, (2) is the chain read as a constraint on families (0.1a): when the energy of a pattern steps, the distances and durations it keeps step with it and its metric changes; a family that changes the total source while holding the metric fixed holds apart what the chain unifies, which is the consensus separation. Second, (2) and (3) constrain the total source, so channels compensate collectively at each point; they do not require every individual energy change to move every metric component, and they become evolution statements only for a specified physical evolution, since s labels a family. Third, changes of units or coordinates are not changes of the physical metric, and a moving support contributes its transport and junction terms $[K_{ab}]$ to the differentiated source.

A frozen-metric family that violates (2) or (3) is not a deformation of the omnium. Whatever it computes is a fact about an auxiliary family on a background, not about a physical configuration.

4 The second variation, and the pure-gauge interpolation

Lemma 8 (constrained second variation). *Write the smooth constraints of $\mathcal{D}_F$ locally as $\mathcal{C}(z) = 0$ on a regular stratum with fixed discrete closure labels, and let an energy functional $\mathcal{E}$ and $\mathcal{C}$ be twice differentiable. Let z_0 be a constrained stationary point with multiplier λ , $D\mathcal{E}(z_0) = D\mathcal{C}(z_0)^*\lambda$. For every twice differentiable admissible curve $z(s)$ through z_0 with $u = \dot{z}(0)$,*

$$\left. \frac{d^2}{ds^2} \mathcal{E}(z(s)) \right|_{s=0} = D^2(\mathcal{E} - \langle \lambda, \mathcal{C} \rangle)[u, u]. \quad (4)$$

Proof. Differentiating $\mathcal{C}(z(s)) = 0$ twice gives $D\mathcal{C}u = 0$ and $D\mathcal{C}\ddot{z} + D^2\mathcal{C}[u, u] = 0$. Then $\frac{d^2}{ds^2} \mathcal{E} = D^2\mathcal{E}[u, u] + D\mathcal{E}[\ddot{z}] = D^2\mathcal{E}[u, u] + \langle \lambda, D\mathcal{C}\ddot{z} \rangle = D^2\mathcal{E}[u, u] - \langle \lambda, D^2\mathcal{C}[u, u] \rangle$. $\square$

Equation (4) carries the response of geometry, source, support and phase together with the curvature of the constraint surface. A Hessian that varies a gauge connection with everything else held fixed is the second variation of a different functional along a different family; it is not (4). Inequality constraints take their own constrained analysis; the discrete closure labels of I-5 are never differentiated, since a change of class is a re-braiding and not a smooth shift. The functional $\mathcal{E}$ is this document's own reading of the objective of the optimisation: the core names none, and the expected behaviour is not necessarily the evolving system's optimum (0.1a); (4) is an identity for that reading. Because no pattern of the omnium is static (Proposition 15), z_0 is a relative equilibrium: (4) is applied to the energy reduced at the conserved quantities, where the gyroscopic terms enter, or to the periodic dynamics directly.

Proposition 9 (the pure-gauge interpolation is not an FDT family). *Let $U : \mathbb{R}^3 \rightarrow \mathbb{G}$ be time-independent and $\theta = U^{-1}dU$ the pure-gauge connection it defines, $d\theta + \theta \wedge \theta = 0$, on a fixed Minkowski background in the temporal gauge, and let $A(s) = s\theta$. Then*

$$F(s) = dA + A \wedge A = s(s-1)\theta \wedge \theta, \quad F'(1/2) = 0, \quad F''(s) = 2\theta \wedge \theta, \quad (5)$$

and the Yang–Mills stress, quadratic in F at fixed metric and coupling, obeys pointwise

$$T_{\mu\nu}(s) = 16s^2(s-1)^2 T_{\mu\nu}(1/2), \quad T'_{\mu\nu}(1/2) = 0, \quad T''_{\mu\nu}(1/2) = -16T_{\mu\nu}(1/2). \quad (6)$$

The family satisfies (2) at $s = 1/2$ and fails (3) there wherever $\theta \wedge \theta \neq 0$; away from $s \in \{0, \frac{1}{2}, 1\}$ it fails (2); its endpoints $s = 0, 1$ are pure-gauge configurations with $T_{\mu\nu} \equiv 0$, the empty state of Proposition 4. It is not a deformation of the omnium at any s .

Proof. $F = s d\theta + s^2 \theta \wedge \theta = (s^2 - s) \theta \wedge \theta$. With $f(s) = s^2(s-1)^2$, $f'(s) = 2s(s-1)(2s-1)$ and $f''(s) = 2(6s^2 - 6s + 1)$, so $f'(1/2) = 0$, $f(1/2) = 1/16$ and $f''(1/2) = -1$; the stress scales as $f(s)/f(1/2)$. Condition (3) with κ fixed requires $\dot{T}_{\mu\nu} = 0$; here $\dot{T}_{\mu\nu}(1/2) = -16 T_{\mu\nu}(1/2) \neq 0$. Condition (2) requires $\dot{T}_{\mu\nu} = 0$; here $\dot{T}_{\mu\nu}(s) \propto f'(s)$, which vanishes only at $s \in \{0, \frac{1}{2}, 1\}$. $\square$

The fixed-metric magnetic energy $E_{YM}(s) \propto f(s)$ has a maximum at $s = 1/2$ along this family, $E''_{YM}(1/2) = -16 E_{YM}(1/2)$. The family never enters the physical domain: wherever $\theta \wedge \theta \neq 0$ its source is nonzero at every $s \notin \{0, 1\}$, which the flat background does not hold (Theorem 2), and its two ends are the empty state (Proposition 4). At $s = 1/2$ the first-order condition (2) holds and the second-order condition (3) fails, so the family fails there even the derivative test that a family of complete solutions with the metric held fixed passes. The negative second derivative is a property of the interpolation on its background. It says nothing about the stability of the omnium: the family is no deformation of the omnium at any s , and the stability of a pattern is the ledger form of I-5b, never the sign of a second variation of a named objective (0.1a). The value $s = 1/2$ is a label on this family and carries no chart; it is not the seam $\alpha_\sigma = 1/2$ of 0.3a.

5 Geometric response and the gap criterion

Lemma 10 (Schur complement). *Let a quadratic form in reduced coordinates (x, y) of one pattern be*

$$q(x, y) = \langle x, Q_{11}x \rangle + 2\operatorname{Re}\langle x, Q_{12}y \rangle + \langle y, Q_{22}y \rangle, \quad Q_{22} \geq dI > 0, \quad (7)$$

with Q_{11} , Q_{22} self-adjoint and Q_{12} bounded. Then

$$\inf_y q(x, y) = \langle x, (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^*)x \rangle, \quad (8)$$

attained at $y = -Q_{22}^{-1}Q_{12}^*x$.

Proof. Complete the square: $q = \langle x, (Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^*)x \rangle + \langle y + Q_{22}^{-1}Q_{12}^*x, Q_{22}(y + Q_{22}^{-1}Q_{12}^*x) \rangle$. $\square$

Free relaxation over the responding coordinates lowers the form by $Q_{12}Q_{22}^{-1}Q_{12}^* \geq 0$ (Lemma 10): a positive frozen-coordinate block decides nothing, and a negative one stays negative. Where the geometry is tied to the source by an explicit constraint, the constrained pullback of Lemma 8 replaces the free minimisation, and its multiplier term $-\langle \lambda, D^2\mathcal{C}[u, u] \rangle$ carries the sign of the curvature of the constraint surface, which is fixed by no general rule: the complete response has no universal sign. Neither form introduces a second substance: x and y are coordinates of one pattern. For operators, (8) requires the domains and boundedness it displays, and static elimination leaves the inertial and gyroscopic terms of the reduced dynamics in place. The energy of these forms is this document's own reading of the objective of the optimisation, which the core leaves unnamed (0.1a), and the stability of a pattern is the ledger form of I-5b: $\langle f | \mathcal{T} | i \rangle = 0$ for every final state f that keeps every exact book. What the forms of this section bound is the spectrum of the generator of a closed pattern's own region, and Proposition 11 turns a uniform positive bound on the complete reduced form into the gap. A frozen-coordinate block bounds that spectrum only along a direction that is itself admissible, and the pure-gauge interpolation is not one (Proposition 9).

Proposition 11 (the gap criterion). *Let $H_{\text{phys}} \geq E_{\text{base}}I$ be the self-adjoint generator of the internal dynamics of a closed pattern's own region, $E_{\text{base}} > 0$ its attained lowest eigenvalue, $H_{\text{rel}} = H_{\text{phys}} - E_{\text{base}}I$, $P_0 = \mathbf{1}_{\{0\}}(H_{\text{rel}})$ the ground projection, and q_{rel} the closed quadratic form of H_{rel} . If, on the whole form domain,*

$$q_{\text{rel}}[\psi] \geq \varepsilon_* \|(I - P_0)\psi\|^2, \quad \varepsilon_* > 0, \quad (9)$$

then $\operatorname{spec}(H_{\text{rel}}) \cap (0, \varepsilon_*) = \emptyset$.

Proof. Let $\psi \neq 0$ lie in the spectral subspace of $(0, \varepsilon_*)$. Then $P_0 \psi = 0$ and

$$q_{\text{rel}}[\psi] = \int_{(0, \varepsilon_*)} \lambda d\|\mathbf{1}_{(-\infty, \lambda]}(H_{\text{rel}}) \psi\|^2 < \varepsilon_* \|\psi\|^2,$$

against (9). □

In FDT the lower edge ε_* is a difference of confinement eigenvalues: $\varepsilon_* = \hbar(\omega_1 - \omega_0) = \hbar\omega_1 - E_{\text{base}}$, the step from the lowest mode cost of the closed pattern's own region, in which the carrier stands, to the next (I-2, I-5), on the closure sector of I-5a; every mode cost is positive (Proposition 15), and the gap is the step between the two lowest. The sector is fixed by the closure constraints and kept by every transition (I-5a, I-5b); it is never chosen to remove low-cost states. The subtraction of E_{base} changes the reference for differences and introduces no empty state: $E_{\text{base}} > 0$ stays a channel of $\mathcal{M}$ (Proposition 4, 0.5). Equality of ε_* with the lowest FDT excitation requires that the observable sector generated from the lowest pattern reaches the lower edge of $\text{spec}(H_{\text{rel}}) \setminus \{0\}$. The operator $D(\Omega_p, B_p; 3, \chi, t)$ of I-5 for the $n = 3$ closures, the class that carries colour (I-5), its domain and boundary data, and the estimate (9) uniform over the admissible closures are to be derived (D-1).

6 No empty region; the infinite volume of the Clay background

Proposition 12 (full support). *Let μ be the energy measure of a configuration of the omnium on its spatial geometry X . Every admissible region $U \subset X$ has $\mu(U) > 0$; in particular*

$$\mu(B_r(x)) > 0 \quad \text{for every } x \in X \text{ and every admissible radius } r \geq \ell_P. \quad (10)$$

Proof. By 0.1b there is only omnium and its patterns, so an admissible region holds omnium, and every length and time in it belongs to a pattern, to a pattern-to-probe measurement or to the floor; the omnium it holds has energy $E = mc^2$ with neither variable vanishing (0.1). □

Equation (10) says that μ has full support at admissible scales: every nonempty admissible region contains energy, and no partition below the floor is invoked. A configuration of the omnium on a geometry of infinite volume is nowhere empty. The Clay vacuum, with $H\Omega_\eta = 0$, assigns zero energy to every region of the same volume.

Theorem 13 (infinite volume). *Let X be connected, complete and of infinite physical volume, and let $\mu(U) \geq e_* > 0$ for every admissible region $U \subset X$. Then X holds pairwise disjoint admissible regions U_j , $j \in \mathbb{N}$, and*

$$\mu(X) \geq \sum_{j=1}^N \mu(U_j) \geq Ne_* \quad \text{for every } N, \quad \text{so } \mu(X) = \infty. \quad (11)$$

Proof. A connected complete geometry of finite diameter is compact (Hopf–Rinow) and of finite volume, so X is unbounded and holds infinitely many points x_j at mutual distances above $d_{\min}$. The balls $U_j = B_{\ell_P}(x_j)$ are pairwise disjoint admissible regions, and monotonicity and finite additivity of μ give (11). □

The hypotheses of Theorem 13 are what the omnium supplies and what full support alone withholds. Full support gives no quantitative floor: the measure $d\mu = e^{-|x|}d^3x$ on $\mathbb{R}^3$ has full support and total mass 8π , its density falling below every positive level; it is not a configuration of the omnium, whose density on admissible regions is bounded below by the density of its spacetime phase. That phase, the $\alpha_\sigma < 1/2$ solid torus, is decompressed omnium—the state toward which compressed omnium relaxes (0.3a, 0.7)—and present wherever the matter phase is not (0.1: one substance in two phases). Its local density ρ_{loc} (0.6) is

positive at every point by 0.1; Theorem 13 uses more: the floor of that phase, a positive infimum ρ_{sp} of ρ_{loc} over the spacetime phase. The floor and its value from the omnium constitutive law are to be derived (D-3). With that floor, every admissible region of physical volume V carries at least $\rho_{\text{sp}}c^2V$, and a region in the matter phase carries more ($\partial\rho_{\text{loc}}/\partial\alpha_\sigma > 0$, 0.6). In the observed domain the standing-wave state of the substrate (0.8) exhibits the floor in the record: its energy density $u = aT^4 = 4.17 \times 10^{-14} \text{ J m}^{-3}$ at $T = 2.7255 \text{ K}$ [9], with $a = 4\sigma_{\text{SB}}/c$, fills the observed domain, its temperature the same in every direction to 1.2×10^{-3} , the dipole [10], and to about 10^{-5} once the dipole is removed [11]. The floor's own window grows with the region. A ball of physical radius R holding at least this density has $\mathcal{M} \geq (4\pi/3)R^3u/c^2$, and on the exterior branch at $r_f = R$ its window obeys

$$\alpha_{\text{ext}}(R) \geq \frac{\kappa u}{3} R^2 = 2.89 \times 10^{-57} \left(\frac{R}{1 \text{ m}} \right)^2, \quad (12)$$

which reaches the seam $\alpha_{\text{ext}} = 1/2$ at $R = 1.32 \times 10^{28} \text{ m}$; the bound itself reaches 1 at $R = 1.86 \times 10^{28} \text{ m}$. A fixed flat background holds no part of this energy (Theorem 2). The regions of (11) are disjoint integration domains within one substrate, their volumes taken in the physical spatial metric, not independent boxes (0.1b); the floor is applied at admissible scales only, never to arbitrarily fine partitions, so no bounded region acquires infinite energy; and a whole excitation extending across several regions is counted once. The record exhibits the floor in the observed domain; over an infinite volume it rests on D-3.

Corollary 14 (no subtraction reaches the Clay vacuum). *On a spatial geometry of infinite volume, the volume the Clay $\mathbb{R}^3$ posits, the energy measure of a configuration of the omnium is infinite (Theorem 13 with the floor of the spacetime phase, D-3), while the Clay vacuum assigns zero energy to every region of its own $\mathbb{R}^3$. No subtraction of a finite constant carries the one to the other: the amount removed from each admissible region would be at least e_* , and its total over the volume is infinite. The FDT gap statement is posed on a closed pattern's own region (Proposition 11), a bounded support within a universal domain (0.8), where E_{base} is finite, H_{rel} is defined, and (9) is posed.*

7 The lowest pattern moves, and carries its network

Proposition 15 (no pattern is static). *Every pattern of the omnium has $\omega_q > 0$ and a finite period $T_q = h/E_q$, with $T_q > t_{\text{min}}$. The lowest pattern of a closed pattern's own region, with energy $E_{\text{base}} > 0$, cycles with period $T_{\text{base}} = h/E_{\text{base}}$.*

Proof. By 0.1a, $E_q = \hbar\omega_q$ and $T_q = 2\pi/\omega_q = h/E_q$; by 0.1 and Proposition 12, $E_q > 0$. A static pattern would have $\omega_q = 0$, hence $E_q = 0$: a vanishing variable of the chain. The thread a pattern runs in the time it keeps, $\ell_q = cT_q$, is a physical stroke, and the floor is the infimum of physical strokes, occupied by none (0.1a); hence $T_q > t_{\text{min}}$ and $E_q < h/t_{\text{min}} = \pi\hbar/t_P$. $\square$

The motion of the lowest pattern is internal: its circulation, support and interacting density structure are the pattern itself (0.1b), and its generator describes them with no external apparatus. The energy accounting of an excitation therefore starts from a moving baseline. If a moving reference pattern $z_0(t)$ and a changed complete pattern $z(t)$ have well-defined conserved total energies, the excess is

$$\Delta E = E_{\text{total}}[z] - E_{\text{total}}[z_0], \quad (13)$$

with the source of any transferred energy, the geometry and every interaction stress included once (0.5, I-7). Circulation redistributes energy among patterns; a transferred amount is not by itself a positive increase of the complete system's energy, and the lower bound the gap needs is a bound on (13) or on its spectral counterpart (9).

For an autonomous generator with $H_{\text{phys}}\Omega_0 = E_{\text{base}}\Omega_0$,

$$e^{-itH_{\text{phys}}/\hbar}\Omega_0 = e^{-itE_{\text{base}}/\hbar}\Omega_0 : \quad (14)$$

the lowest pattern stays in its eigenspace while its phase advances by 2π in $T_{\text{base}} = h/E_{\text{base}}$. That phase is the helical phase of I-4, and T_{base} is the time the pattern keeps (0.1a): (14) is the motion of Proposition 15 written in the spectral representation. The Clay convention $H\Omega_\eta = 0$ sets this period to infinity—a pattern that never completes a cycle, which is no pattern of the omnium. For suitable observables the correlations $\langle O\Omega_0, e^{-itH_{\text{rel}}/\hbar}O\Omega_0 \rangle$ depend on t and encode the accessible mode costs; their lower edge is ε_* .

Density contrasts and the density-geodesic network

By 0.1, around every point a sphere exists containing both higher- and lower-density regions, so interaction pairs are available there and I-1 has no force-free point. Let $\mathcal{R}_F$ be the admissible regions and ρ_E an energy-density reading where one is defined. The precise form is

$$\forall x \exists B_x \in \mathcal{R}_F : \quad x \in B_x, \quad \text{ess sup}_{B_x} \rho_E > \text{ess inf}_{B_x} \rho_E, \quad (15)$$

the essential extrema taken in the physical volume measure and compared with each other, not with a value at the centre. Equation (15) uses admissible regions only.

Because its density is nowhere uniform, read between neighbouring parts of its own extent as in (15) (I-3a), the lowest pattern carries the nodal architecture of P-2: the loci where the transverse density gradient vanishes are its density geodesics, aligned channels into which excitations lock and along which the pattern's energy is transported, sustained by the motion of Proposition 15. The lowest pattern is one coupled configuration: its helical geodesic (I-4), its density-geodesic network (P-2), its geometry (I-3) and its circulation. The channels are formed by the same omnium; they are not an independently supplied support (0.1b).

The density contrast and the transport law belong together. The excess (13) compares two complete configurations, each with its own sustaining network; changing the helix while freezing or omitting the network computes the energy of no physical configuration. Where a complete total-energy density is valid and the integrals are finite,

$$\Delta E = \int_{\Sigma_z} \rho_E[z] dV_{h[z]} - \int_{\Sigma_0} \rho_E[z_0] dV_{h[z_0]}, \quad (16)$$

each density and each volume element belonging to its own complete solution. General relativity supplies no unique local gravitational energy density, and in FDT the binding energy is read from the solved geometry and never added as a local term (0.5, I-7); where no density representation exists, the comparison uses the complete energy functional or H_{rel} . Higher density in one part accompanies lower density elsewhere, so an internal contrast is never integrated as a net increase with the compensation omitted. The self-consistent transport law of the lowest pattern and its network—phase return and integral closure included—is to be derived (D-2).

8 The Clay boundary, stated

The Clay statement is posed where $R_s = 0$: a flat background at every field radius and a vacuum of zero total source. No FDT configuration attains it: the exterior branch approaches $\alpha_{\text{ext}} \rightarrow 0^+$ and never arrives (I-6), and with the floor of the spacetime phase (D-3) every configuration carries at least e_* on every admissible region, so no sequence of configurations converges to the zero-energy vacuum. It is the endpoint FDT excludes, reached by ignoring the response of geometry to energy—by ignoring general relativity. If general relativity is ignored, the Clay problem statement does not apply to reality (Theorem 5).

Where FDT, general relativity and quantum mechanics converge, FDT reproduces their results; where FDT declines their assumptions—here the fixed background and the empty vacuum—it is under no obligation to reproduce them. A Clay requirement with no representation in nature’s domain is not addressed by an FDT proof; it is declined. The Clay statement poses pure Yang–Mills for every compact simple gauge group: the carrier with no closure. In FDT the carrier stands only inside a closed pattern’s own region (I-2), and colour is the $SU(3)$ of the internal nodal surface of the $n = 3$ closures (I-5). What FDT states in place of the Clay statement is the physical gap problem, posed there: the generator of the internal dynamics of a closed pattern’s own region, on the closure sector, with the uniform estimate (9) on the coupled domain of complete patterns, where every variation carries its geometric response (Theorem 7, Lemma 8) and its network (§7). Its lower edge is a difference of confinement eigenvalues (I-5), and its derivation is D-1.

The distributional short-distance behaviour a Wightman theory requires is not a physical density divergence: FDT has none (0.8), and its floor $2\ell_P$, $2t_P$ bounds every stroke below (0.1a). Fixed-background well-posedness (Remark 6) and the spectral edge are different questions; the present argument settles the domain on which the second is posed.

9 Ledger and convergence

Derived here from core v6.3 and displayed mathematics: Theorem 2, Corollary 3, Proposition 4 and Theorem 5 (the fixed background and the empty state); Theorem 7 (compatibility); Lemma 8 and Proposition 9 (the second variation and the pure-gauge interpolation); Lemma 10 and Proposition 11 (response and the gap criterion); Propositions 12 and 15 (full support; no static pattern); Theorem 13 and Corollary 14 under their displayed hypotheses, with the floor’s window (12); (13) and (16) as the bookkeeping of complete patterns, (14) as the spectral form of Proposition 15, and (15) as the density contrast of 0.1 on admissible regions.

To be derived.

id	item	binds
D-1	the confinement operator $D(\Omega_p, B_p; 3, \chi, t)$ of the $n = 3$ closures, the class that carries colour, its domain and boundary data, and the estimate (9) uniform over the admissible closures, with the observable sector generated from the lowest pattern reaching the lower edge	I-5, I-2; the core’s O-9, and O-6 for the gluon octet’s dynamics
D-2	the self-consistent transport law of the lowest pattern with its density-geodesic network, phase return and integral closure, and the coupled evolution of geometry and source it belongs to	P-2, I-3; the core’s O-13 for junction data
D-3	the density floor $\rho_{sp} > 0$ of the spacetime phase and its value from the omnium constitutive law, and with it the value e_* of (11)	0.3a, 0.6, 0.7, 0.8; the core’s O-4, O-14, and the standing-wave half of O-15

Convergence. For the same source and data FDT and general relativity solve the same equation and return the same curvature (I-3). A laboratory Yang–Mills configuration of energy 1 GeV and extent 1 fm has $\mathcal{M} = 1.78 \times 10^{-27}$ kg, $R_s = 2G\mathcal{M}/c^2 = 2.65 \times 10^{-54}$ m and $\alpha_{\text{ext}}(1 \text{ fm}) = 2.65 \times 10^{-39}$; at that order the coupled equations reduce to the fixed-background equations, and the classical well-posedness theorems [4, 5] are recovered as the $\alpha_{\text{ext}} \rightarrow 0^+$ reading of the coupled solution, with the geometric response carried as a correction of relative size α_{ext} . The masses of closed patterns are the confinement eigenvalues of D-1, whose operator is the core’s register item O-9: the glueball spectra computed in lattice gauge theory [6, 7] are tests for it, never its outputs. The measured record, a hadron spectrum with no massless strongly interacting state [8], is accepted in full; FDT and the consensus agree on every measured value here. What FDT declines is the endpoint $R_s = 0$: the exactly flat background and the exactly empty vacuum of the problem statement, the endpoint of the one-sided limit $\alpha_{\text{ext}} \rightarrow 0^+$, which no configuration attains (I-6).

Provenance. FDT core: CORE-FDT-INVARIANTS.md, locked v6.3, author lock 2026-10-26, SHA-256 1989c0b4f7f1471f4eaa9305d231a06a9bbe4e6f07b4560b223f412e16b5463d. All core quotations are by section number.

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